On Polarization Types and Monodromy of Lagrangian Fibrations

Von der Fakultät für Mathematik und Physik
der Gottfried Wilhelm Leibniz Universität Hannover
zur Erlangung des Grades

Doktor der Naturwissenschaften
Dr. rer. nat.

genehmigte Dissertation
von

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geboren am 12. Juli 1986 in Hamburg

2016

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Tag der Promotion: 11. Juli 2016

Kurzzusammenfassung

Die generische Faser einer Lagrangefaserung auf einer irreduziblen holomorph symplektischen Mannigfaltigkeit ist eine abelsche Varietät. Wir orden jeder Lagrangefaserung einen Polarisierungstyp zu, welcher von einer Polariserung auf einer generischen Faser kommt. Es folgt, dass der Polarisierungstyp konstant in Familien von Lagrangefaserungen ist. Desweiteren wird der Polarisierungstyp von $K3^{[n]}$–typ und verallgemeinerten Kummerfaserungen bestimmt. Für den $K3^{[n]}$–Fall ist der Typ immer prinzipal. Der verallgemeinerte Kummerfall zeigt, dass der Polarisierungstyp im Allgemeinen von der Zusammenhangskomponente des Modulraums der Lagrangefaserungen abhängt. Die Berechnung benutzt Modulraumtheorie von Lagrangefaserungen, welche in Form von Monodromieinvarianten Funktionen auf der Menge der isotropen Klassen in der zweiten ganzzahligen Kohomologie auftritt. Solch eine Monodromieinvariante wird für den verallgemeinerten Kummerfall konstruiert.

Schlagwörter: Irreduzible holomorph symplektische Mannigfaltigkeit, Hyperkähler Mannigfaltigkeit, Lagrangefaserung, Polarisierungstyp, Monodromieinvariante

Abstract

The generic fiber of a Lagrangian fibration on an irreducible holomorphic symplectic manifold is an abelian variety. Associate a polarization type to such Lagrangian fibrations coming from polarizations on a generic fiber. It follows that this polarization type is constant in families of Lagrangian fibrations. Further, we determine the polarization type of $K3^{[n]}$–type and generalized Kummer fibrations. For the $K3^{[n]}$ case, the type is always principal. The generalized Kummer case shows, that in general the polarization type depends on the connected component of the moduli space of Lagrangian fibration. The computation involves moduli theory of Lagrangian fibrations, which appears in the form of a monodromy invariant function on the set of isotropic classes on the second integral cohomology. Such a monodromy invariant is constructed for the generalized Kummer case.

Keywords: Irreducible holomorphic symplectic manifold, hyperkähler manifold, Lagrangian fibration, polarization type, monodromy invariant

Contents

Introduction

The natural complex or differential geometrical generalization of smooth projective varieties are compact Kähler manifolds. By the famous decomposition Theorem of A. Beauville and F. Bogomolov, cf. Theorem 1.1.5, which was first proven by S. Kobayashi, a Ricci flat compact Kähler manifold splits up to an étale cover, into three types of manifolds, namely a torus, Calabi–Yau manifolds and irreducible holomorphic symplectic manifolds.

In this thesis we are interested in the latter type of manifolds i.e. irreducible holomorphic symplectic ones, which carry a unique holomorphic symplectic form, up to a scalar. Yau's solution of the Calabi conjecture ensures the existence of Riemannian metrics on them with holonomy group exactly isomorphic to the special unitary group and that is why such manifolds are also called (compact) *hyperkähler*.

The geometry of irreducible holomorphic symplectic manifolds seems to be quite rigid since very few deformation types are known. In dimension two, these manifolds are nothing but K3 surfaces i.e. we can see irreducible holomorphic symplectic manifolds as higher dimensional generalizations of them. The first higher dimensional example was found by A. Fujiki [**Fuj83**] which is of dimension four. A. Beauville [**Bea84**] generalized this result and constructed two infinite series of hyperkähler manifolds for each dimension $2n$, which are today known as the Hilbert scheme of n points of a K3 surface and generalized Kummer manifolds. The latter one is constructed with a two–torus and is a generalization of the classical Kummer K3 surface. K. O'Grady [**O'G99**], [**O'G03**] found two more exceptional examples in dimension ten and six which are certain resolutions of moduli spaces of sheaves on a K3 or abelian surface, respectively. Up to deformation, these are all known examples.

As higher dimensional generalization of K3 surfaces, irreducible holomorphic symplectic manifolds and K3 surfaces share many similar properties. For instance, the second cohomology $H^2(X,\mathbb{Z})$ of any irreducible holomorphic symplectic manifold X admits the well known *Beauville–Bogomolov–Fujiki* quadratic form $(\cdot,\cdot)$ which is non–degenerate and of signature $(3, b_2(X)-3)$. In the case of a K3 surface this is nothing but the usual intersection product for surfaces. This leads to lattice theoretical methods.

Also there is a Local Torelli by A. Beauville [**Bea84**] and Verbitsky's celebrated Global Torelli [**Ver13**], see Theorem 1.3.4 and Theorem 1.3.8, respectively.

More specifically we are interested in *Lagrangian fibrations* on such manifolds. The only possible nontrivial fibrations irreducible holomorphic symplectic manifolds can admit are Lagrangian as D. Matsushita showed, see Theorem 2.1.3. Lagrangian fibrations help us to understand the geometry of such manifolds. It is hoped that Lagrangian fibrations will be useful for the classification of irreducible holomorphic symplectic manifolds, see for instance [**Saw03**].

Results

One of the guiding research questions at the beginning of the author's PhD studies concerned the possible geometry of the smooth fibers of a Lagrangian fibration.

Let $f : X \to B$ be a Lagrangian fibration. It is well known that all smooth fibers are abelian varieties, even if X is not projective, cf. Proposition 2.1.2 and Theorem 2.1.3. For an abelian variety F of dimension $\dim F = n$, there is a well known classical notion of a *polarization*, cf. [**BL03**, p. 70], which is by definition the first Chern class $H = c_1(L)$ of an ample line bundle L of F. Often one calls the ample line bundle L a polarization. Furthermore, one can associate to such a polarization a *type*, which is a tuple

$$\underline{\mathrm{d}}(L) \;=\; (d_1, \ldots, d_n)$$

of positive integers such that d_i divides d_{i+1}, cf. section B.1.

Given a smooth fiber F of the Lagrangian fibration f which is an abelian variety as mentioned above, an immediate and interesting question is to ask for polarizations and their types on it.

Question I.1 *Are there any restrictions on the possible types of induced polarizations from X on the smooth fibers of a Lagrangian fibration $f : X \to B$?*

The author has found an answer to this question in the following sense.

First of all, it is not clear, how to obtain a polarization on a smooth fiber F of the Lagrangian fibration $f : X \to B$ if X is not projective. However, due to the following statement, which is related to an observation of C. Voisin [**Cam06**, Prop. 2.1], it is always possible.

Proposition I.2 (Proposition 3.1.3) *For any smooth fiber F there is a Kähler class ω on X such that the restriction $\omega|_F$ is integral and primitive.*

Such a class ω is called *special Kähler class* (with respect to F) and defines a polarization $\omega|_F$ on the abelian variety F in the sense above. This polarization one can associate its type $\underline{\mathrm{d}}(\omega|_F) \coloneqq (d_1, \ldots, d_n)$ where again d_i are positive integers such that d_i divides d_{i+1}.

Definition I.3 (Section 3.4) The *polarization type* of a Lagrangian fibration $f : X \to B$ is

$$\underline{\mathrm{d}}(f) \;\coloneqq\; \underline{\mathrm{d}}(\omega|_F) \;=\; (d_1, \ldots, d_n)\,.$$

This definition seems to be a bit ad–hoc, but it is convenient for the introduction. The first main result is the following.

Theorem I.4 (Chapter 3) *Let $f : X \to B$ be a Lagrangian fibration with $\dim X = 2n$. Then the following statements hold.*

(*i*) (Proposition 3.4.1) *The polarization type $\underline{\mathrm{d}}(f)$ is well defined i.e. does not depend on the chosen smooth fiber and the chosen special Kähler class (with respect to this fiber) and is a primtive vector in $\mathbb{Z}^n$.*

(*ii*) (Theorem 3.4.3) *The polarization type is a deformation invariant of the fibration i.e. if $f' \colon X' \to B'$ is a Lagrangian fibration deformation equivalent to f*[1]*, then $\underline{\mathrm{d}}(f) = \underline{\mathrm{d}}(f')$.*

(*iii*) (Proposition 3.3.3, Proposition 3.4.4) *Let B° denote the subset of B which parametrizes the smooth fibers. Then there exists a* family of special Kähler classes*, that is a map $\alpha : B^\circ \to \mathcal{H}$ where $\mathcal{H} \subset (R^2\pi_\star\mathbb{Z} \otimes \mathcal{O}_B)|_{B^\circ}$ is a subbundle and $\alpha(t)$ is a special Kähler class with respect to the smooth fiber X_t for every $t \in B^\circ$. In particular $\underline{\mathrm{d}}(\alpha(t)) = \underline{\mathrm{d}}(f)$ for every $t \in B^\circ$.*

(*iv*) (Corollary 3.5.3) *The family of special Kähler classes α induces a holomorphic map, called* moduli map*,*

$$\begin{aligned} \phi \ : \ B^\circ &\longrightarrow \mathcal{A}_{\underline{\mathrm{d}}(f)}\,, \\ t &\longmapsto (X_t, \alpha(t)) \end{aligned}$$

where $\mathcal{A}_{\underline{\mathrm{d}}(f)}$ denotes the moduli space of $\underline{\mathrm{d}}(f)$ polarized abelian varieties.

The next main result is the computation of the polarization type of Lagrangian fibrations of $\mathrm{K3}^{[n]}$–type.

Theorem I.5 (Theorem 5.4.1) *Let $f : X \to \mathbb{P}^n$ be a Lagrangian fibration of $K3^{[n]}$–type. Then we have for its polarization type*

$$\underline{\mathrm{d}}(f) \ = \ (1, \ldots, 1)\,.$$

Note that Theorem I.4 and Theorem I.5 are a part of the author's paper [**Wie15**], which has a similar title.

The proof of Theorem I.5 involves moduli theory of Lagrangian fibration. We will explain this later for the generalized Kummer type case.

After finding the result above, the author stated the following question.

Question I.6 (Conjecture 3.4.7) Let $f_i : X_i \to B_i$, $i = 1, 2$, be two Lagrangian fibrations such that X_1 and X_2 are deformation equivalent. Then their polarization types coincide

$$\underline{\mathrm{d}}(f_1) \ = \ \underline{\mathrm{d}}(f_2)\,.$$

The computation of the polarization type of a Lagrangian fibration of generalized Kummer type, which is the next main result, shows, that the answer to this question is negative.

[1] This means we have an S–morphism $\phi \colon \mathcal{X} \to P$ such that S is a connected complex space with finitely many irreducible components, $\mathcal{X} \to S$ is a family of irreducible holomorphic symplectic manifolds and $P \to S$ is a family of projective varieties such that $\phi_t \coloneqq \phi|_{\mathcal{X}_t} \colon \mathcal{X}_t \to P_t$ is a Lagrangian fibration for all $t \in S$ and there are points $t_i \in S$, $i = 1, 2$, such that $f = \phi_{t_1}$ and $f' = \phi_{t_2}$.

Theorem I.7 (Theorem 5.4.1, Proposition 5.3.1) *Let $f : X \to \mathbb{P}^n$ be a Lagrangian fibration of generalized Kummer type. If $d = \operatorname{Div}(\lambda)$ denotes the divisibility*[2] *of $\lambda = c_1(f^\star \mathcal{O}_{\mathbb{P}^n}(1))$, then d^2 divides $n+1$ and we have for the polarization type*

$$\underline{\mathrm{d}}(f) \;=\; \left(1, \ldots, 1, d, \frac{n+1}{d}\right) .$$

Further, for a fixed dimension $\dim X = 2n$, the divisibilities of classes λ as above which can appear for the generalized Kummer type, are exactly the positive integers d such that d^2 divides $n+1$.

The proofs of the results Theorem I.5 and Theorem I.7 above are similar and involve moduli theory of Lagrangian fibrations of $\mathrm{K3}^{[n]}$ and generalized Kummer type, as for instance exploited in [**Mar14**]. The moduli theory appears in form of what is called a *monodromy invariant*.

Let X be an irreducible holomorphic symplectic manifold and consider the monodromy group $\operatorname{Mon}^2(X)$, see subsection 1.5.1. A *faithful monodromy invariant*, see section 5.1 and [**Mar13**, Def. 5.16], is a $\operatorname{Mon}^2(X)$–invariant map $\vartheta : I(X) \to \Sigma$ where $I(X) \subset H^2(X, \mathbb{Z})$ is a $\operatorname{Mon}^2(X)$–invariant subset and Σ is an arbitrary set, such that the induced map $I(X)/\operatorname{Mon}^2(X) \to \Sigma$ is injective.

The following is a generalized Kummer analogue of E. Markman's monodromy invariant for the $\mathrm{K3}^{[n]}$ case, see [**Mar14**, 2.].

Let X be of generalized Kummer type. For a fixed positive integer d, let denote $I_d(X) \subset H^2(X, \mathbb{Z})$ the set of all primitive isotropic classes with divisibility d. For the case that d^2 divides $n+1$, let $\Sigma_{n,d}$ denote the set of isometry classes of pairs (H, w) such that H is a lattice isometric to the lattice $L_{n,d}$ which is defined in (5.2.4) and $w \in H$ is a primitive class with $(w, w) = 2n+2$. The following main result is needed for Theorem I.7.

Theorem I.8 (Section 5.2.5, Theorem 5.2.9) *Let X be a generalized Kummer type manifold of dimension $2n$ and d a positive integer such that d^2 divides $n+1$. There is a surjective faithful monodromy invariant*

$$\vartheta \;:\; I_d(X) \longrightarrow \Sigma_{n,d}$$

of the manifold X.

By E. Markman, we also have a monodromy invariant for the $\mathrm{K3}^{[n]}$ case which is defined similar to the generalized Kummer case, see section 5.2.10 or [**Mar14**, 2.]. In the following we denote the monodromy invariant for the $\mathrm{K3}^{[n]}$ case also by ϑ.

The monodromy invariant seems to be quite technical and in order to give an geometric interpretation, we state the following partial and minor results. Indeed, the monodromy invariant can detect to which connected component of the moduli a Lagrangian fibration belongs to as explained in the next statements.

[2]Here we mean with the divisibility $k = \operatorname{Div}(\lambda)$, the largest positive number k, such that $(\lambda, \cdot)/k$ is an integral form.

First we give the following statement, which can be obtained with use of results in [**Mar13**] and [**Mat13**].

Theorem I.9 (Theorem 2.4.7) *Let λ be a primitive and isotropic element in the $K3^{[n]}$ or generalized Kummer lattice Λ and fix a connected component $\mathfrak{M}_\Lambda^\circ$ of the moduli of Λ–marked pairs. There exists a non–Hausdorff, connected complex submanifold $\mathfrak{U}_{\lambda^\perp}^\circ$ of codimension one*[3] *in $\mathfrak{M}_\Lambda^\circ$ with the following properties.*

- (i) (Theorem 2.4.7) *It parametrizes isomorphism classes of marked pairs (X, η) of $\mathfrak{M}_\Lambda^\circ$ with X of $K3^{[n]}$ or generalized Kummer type, respectively, admitting a Lagrangian fibration $f : X \to \mathbb{P}^n$ such that*

$$\eta\left(c_1\left(f^\star\mathcal{O}_{\mathbb{P}^n}(1)\right)\right) \;=\; \lambda\,.$$

- (ii) (Proposition 3.4.6) *The associated Lagrangian fibrations of two marked pairs which define points in $\mathfrak{U}_{\lambda^\perp}^\circ$ have the same polarization type.*

We refer to this space $\mathfrak{U}_{\lambda^\perp}^\circ$ as a *connected component of the moduli of Lagrangian fibrations*.

We can state the geometric interpretation of the monodromy invariant.

Proposition I.10 (Proposition 2.4.8, [**Mar13**, Lem. 5.17]) *Let $f_i : X_i \to \mathbb{P}^n$, $i = 1, 2$, denote two Lagrangian fibrations with both of $K3^{[n]}$–type or both of generalized Kummer type. Accordingly, let Λ denote the $K3^{[n]}$–lattice or the generalized Kummer lattice respectively and set $L_i \coloneqq f_i^\star\mathcal{O}_{\mathbb{P}^n}(1)$. Then the following statements are equivalent.*

- (i) *The Lagrangian fibrations f_i are deformation equivalent.*
- (ii) *There exist markings $\eta_i : H^2(X_i, \mathbb{Z}) \to \Lambda$ such that the marked pairs (X_i, η_i) are contained in the same connected component $\mathfrak{U}_{\lambda^\perp}^\circ$ for a primitive isotropic class λ in the $K3^{[n]}$ or generalized Kummer lattice.*
- (iii) [**Mar13**, Lem. 5.17] *We have $\mathrm{Div}(c_1(L_1)) = \mathrm{Div}(c_1(L_2))$ for the corresponding divisibilities and $\vartheta(c_1(L_1)) = \vartheta(c_1(L_2))$ for the monodromy invariant.*

Note that (iii) in the above Proposition is a general property of monodromy invariants proven in [**Mar13**, Lem. 5.17].

Structure of the thesis

In **Chapter 1** we begin with the summary of the general theory of irreducible holomorphic symplectic manifolds. We give the standard definitions, examples and results. In particular, we discuss the Beauville–Bogomolov–Fujiki quadratic form, the Torellis, orientation and monodromy results.

Chapter 2 has the same purpose as Chapter 1, but we deal with important facts about Lagrangian fibrations. In section 2.2 we explain the close relation between isotropic, nef line bundles and Lagrangian fibrations, in particular [**Mat13**] is important for this section.

[3] That is $\dim \mathfrak{U}_{\lambda^\perp}^\circ = 20$ for the $\mathrm{K3}^{[n]}$ and $\dim \mathfrak{U}_{\lambda^\perp}^\circ = 4$ for the generalized Kummer case.

In section 2.3 and 2.4 the moduli theory of Lagrangian fibrations is explained which is mostly a recollection of known facts for the convenience of the reader. In the case of $\mathrm{K3}^{[n]}$–type or generalized Kummer fibrations, this relies on methods developed by E. Markman in [**Mar11**] and [**Mar14**]. Besides that two results of D. Matsushita [**Mat09**], [**Mat13**] play an important role. The former one, see Theorem 2.4.1, states that every Lagrangian fibration can be considered as a member of a family of Lagrangian fibrations parametrized by a small representative of deformation space $\mathrm{Def}(X, L)$ of the pair (X, L), where L is the pullback of an ample line bundle on the base space. Using this theory, we describe how to obtain a connected component of the moduli space of $\mathrm{K3}^{[n]}$–type or generalized Kummer fibrations, as stated in Theorem I.9.

Chapter 3 is the core of the thesis and the first sections have the purpose to construct the polarization of a Lagrangian fibration and prove the properties as stated in Theorem I.4. In particular, the relation between the geometry of the moduli of Lagrangian fibrations and the polarization type as stated in Theorem I.9 is given, compare for instance Propostion 2.4.8 and Theorem 3.4.6. The chapter ends with a remark on a generalization of Matsushita's conjecture.

Chapter 4 has two main purposes. The first, see section 4.3, is to construct a canonical monodromy invariant $\mathrm{O}(\widetilde{\Lambda})$–orbit of primitive isometric embeddings $\Lambda \hookrightarrow \widetilde{\Lambda}$ where Λ is the generalized Kummer lattice and $\widetilde{\Lambda} = \mathrm{U}^{\oplus 4}$ is the Mukai lattice associated to an abelian surface. This orbit, as explained above, is a main ingredient for the construction of the monodromy invariant in the next Chapter 5. G. Mongardi's monodromy result, see Theorem 1.5.12, is significant.

The second, see section 4.4, is the introduction of Beauville–Mukai systems which are examples of Lagrangian fibrations. Further their polarization type is determined. The computation uses results from a work of C. Ciliberto and G. van der Geer [**CvdG92**], see also Appendix B.3. Beauville–Mukai systems play an important role for the main results Theorem I.5 and Theorem I.7, see the summary of Chapter 5. Examples of Lagrangian fibrations on the O'Grady manifolds are given in section 4.5.

Since everything is related to the moduli theory of sheaves on projective holomorphic symplectic surfaces, the chapter starts with an introduction to this topic.

Chapter 5 deals with the construction of the monodromy invariant as stated in Theorem I.8, see section 5.2. Section 5.3 shows with use of the monodromy invariant, that in every connected component of the moduli of generalized Kummer type fibrations there is a Beauville–Mukai system. This is an analogy of the $\mathrm{K3}^{[n]}$ case [**Mar14**, 3., Ex. 3.1]. Finally in section 5.4, we compute the polarization types as stated in Theorem I.5 and Theorem I.7.

There is an **Appendix**. The first part A deals with some basic definitions from lattice theory. In this work, lattice theory is frequently used and if the reader is not familiar with a lattice theoretical notion we refer to Appendix A.

The second part B are basic definitions and results from the theory of abelian varieties. In particular the classical definition of polarizations and their types are given in section B.1. However, section B.2, in particular subsection B.2.3, deals with complementary abelian subvarieties and Proposition B.2.14 is crucial for the computation of the polarization type of Beauville–Mukai systems of generalized Kummer type in subsection 4.4.15. The last section B.3 exploits the paper [**CvdG92**]. The statements in this section deal with Picard numbers of certain abelian varieties which are important for the computation of the polarization types in Chapter 5.

Acknowledgements

First of all, I thank my advisor Klaus Hulek for the supervision of my thesis and accepting me as a PhD student without knowing much about me. In particular, for the many discussion we had and the invitation to the workshop *Modular Forms and Geometry of Modular Varieties* at the IMPA in Rio de Janeiro, May 2015 and the opportunity to give a talk about my project.

I thank Christian Lehn for several discussions and the suggestion to actually begin to consider polarization types, beside being suggested by Question I.1, at the fall school *Around Torelli's theorem for K3 surfaces* at the IRMA in Strasbourg, October/November 2013.

Especially, I want to thank Eyal Markman for hospitality at the University of Massachusetts in Amherst in fall 2014 and explaining me so much about the moduli of Lagrangian fibrations. During the stay, an essential part of the paper [**Wie15**] was finished.

The algebraic geometry group, i.e. the *Institut für Algebraische Geometrie*, in Hannover was very helpful, inspiring and the atmosphere was always nice. In particular, I thank Malek Joumaah and Víctor González-Alonso with whom I had several helpful discussions about many different mathematical topics.

Further, I have to thank Malte Wandel and Giovanni Mongardi for answering me some specific questions and also the referee of the paper [**Wie15**], who read it carefully and pointed out several mistakes and improvements. I thank Olivier Debarre for pointing me out the idea of the proof of Lemma B.2.9.

I gratefully acknowledge the support of the DFG Graduiertenkolleg 1463 "Analysis, Geometry and String Theory".

There are several colleagues who made my time at the Leibniz Universität more enjoyable – thanks for that goes especially to my former office mates Maike Tormählen, Felix Lubbe, Marcus Sperling, Jakob Geipel and Frederik Tietz. But also to Sarah Blanke, Elisabeth Werner and the Italians Fabio Apruzzi, Davide Veniani and Roberto Laface. We had always a good time at the *Sprengelstube*, at conferences or during other activities.

Last but not least, I thank my parents and my sister Jasmin for the support during my studies in Hamburg and during my time in Hannover and to encourage me to do a PhD.

CHAPTER 1

Hyperkähler Manifolds

This chapter has the purpose of giving an overview of the theory of irreducible holomorphic symplectic manifolds, also called hyperkähler manifolds. The main references are the well known book chapter [**GHJ03**] by D. Huybrechts, A. Beauville's famous paper [**Bea84**], K. O'Grady's lecture notes [**O'G14b**], [**O'G14a**] from GAeL 2014 in Trieste and E. Markman's survey paper [**Mar11**].

1.1. Holomorphic symplectic manifolds

Definition 1.1.1 A *holomorphic symplectic form* on a complex manifold X is a closed and everywhere non–degenerate holomorphic two–form σ. The pair (X, σ) is called *holomorphic symplectic manifold.*

The existence of such a form σ has several immediate consequences.

Remark 1.1.2 Let (X, σ) be a holomorphic symplectic manifold.

(i) For a point $x \in X$ the form σ_x is a non–degenerate and alternating form on the holomorphic tangent space $\mathcal{T}_{X,x}$. Therefore the manifold X is of even complex dimension $\dim_{\mathbb{C}} X = 2n$.

We will denote with $2n$ the dimension of such manifolds in this thesis.

(ii) The form $\sigma^n := \sigma^{\wedge n} := \sigma \wedge \cdots \wedge \sigma$ gives a nowhere vanishing holomorphic section of the canonical bundle $K_X = \Omega_X^{2n}$, namely a volume form. We conclude that $K_X \cong \mathcal{O}_X$ is trivial and hence X has vanishing first Chern class $c_1(X) = 0$.

(iii) Further the non–degeneracy gives an isomorphism

$$\mathcal{T}_X \longrightarrow \Omega_X^1\,, \quad Z \mapsto \sigma(Z, \cdot)$$

between the holomorphic tangent bundle and the bundle of holomorphic one–forms.

In this thesis we are interested in a special kind of holomorphic symplectic manifolds.

Definition 1.1.3 A complex manifold X is called *irreducible holomorphic symplectic* if

(i) X is compact Kähler,

(ii) X is simply connected,

(iii) the space $H^0(X, \Omega_X^2)$ of holomorphic two–forms is generated by a nowhere degenerate holomorphic two–form σ.

Whe have the following immediate implications.

Remark 1.1.4 Let X be an irreducible holomorphic symplectic manifold.

(i) Since X is compact Kähler, the holomorphic two–form σ is automatically closed (use the Kähler identities), hence it is a holomorphic symplectic form. Therefore everything in Remark 1.1.2 applies for X.

(ii) In particular, the Hodge decomposition holds. As $H^0(X,\Omega_X^2) = H^{2,0}(X) = \mathbb{C}\sigma$, we have

$$H^2(X,\mathbb{C}) \;=\; \mathbb{C}\cdot\sigma \oplus H^{1,1}(X,\mathbb{C}) \oplus \mathbb{C}\cdot\overline{\sigma}\,.$$

More general, the space of holomorphic p–forms is given by

$$H^0(X,\Omega_X^p) \;=\; \begin{cases} \mathbb{C}\cdot\sigma^{p/2}, & \text{if } p \text{ is even,} \\ \quad 0, & \text{if } p \text{ is odd,} \end{cases}$$

see [**Bea84**, Prop. 3].

(iii) Simply connectivity implies $H^1(X,\mathbb{Z}) = 0$, therefore by Hodge decomposition we have $H^1(X,\mathcal{O}_X) = H^0(X,\Omega_X^1) = 0$. This implies that $c_1 : \mathrm{Pic}(X) \to \mathrm{NS}(X)$ is an isomorphism by using the long exact sequence of the exponential short exact sequence.

One of the main motivations to study irreducible holomorphic symplectic manifolds is the following well known Beauville–Bogomolov decomposition theorem [**Bog74**], [**Bea84**, Thm. 1], which was first proven by S. Kobayashi [**Kob81**].

Theorem 1.1.5 (Beauville–Bogomolov–Kobayashi) *Let X be a compact Kähler manifold X with vanishing first Chern class $c_1(X) = 0$ in the real cohomology $H^2(X,\mathbb{R})$*[1]*. Then X admits a finite étale covering $\widetilde{X} \to X$ such that $\widetilde{X}$ is biholomorphic to a product of the form*

$$\widetilde{X} \;\cong\; T \times \prod_i Y_i \times \prod_i Z_i$$

such that

- *T is a complex torus,*
- *Y_i are Calabi–Yau manifolds in the strict sense* i.e. Y_i is compact Kähler with trivial canonical bundle $K_{Y_i} \cong \mathcal{O}_{Y_i}$ such that $h^{p,0}(Y_i) = 0$ for $1 < p < \dim Y_i$,
- *and Z_i are irreducible holomorphic symplectic.*

In contrast to Calabi–Yau manifolds, there are *by far less*[2] examples of families of irreducible holomorphic symplectic manifolds known. In the following, we give the common examples.

[1] For compact Kähler manifolds, vanishing first Chern class is equivalent for the manifold being Ricci flat.

[2] This is meant metamathematically.

1.1.6. K3 surfaces. In dimension two, an irreducible holmorphic symplectic manifold is nothing but a K3 surface. Recall that a K3 surface S is a compact complex smooth surface such that the canonical bundle $K_S \cong \mathcal{O}_S$ is trivial and $H^1(S, \mathcal{O}_S) = 0$.

1.1.7. Douady spaces of points. To construct the most easiest higher dimensional example one starts with a K3 surface S. The *Douady space* $S^{[n]}$ *of n points* is the complex space which parametrizes zero–dimensional subspaces of S of length $l(Z) := \dim_{\mathbb{C}} \mathcal{O}_Z(Z) = n$. A general point Z of $S^{[n]}$ is therefore of the form

$$Z \;=\; \{z_1, \ldots, z_n\}$$

with points $z_1, \ldots, z_n \in S$. A. Beauville [**Bea84**] showed that $S^{[n]}$ is an irreducible holomorphic symplectic manifold of dimension $2n$ with Betti number $b_2(S^{[n]}) = 23$.

Consider the *n–th symmetric product* $S^{(n)} := \operatorname{Sym}^n S := (S \times \cdots \times S)/\mathfrak{S}_n$ where $\mathfrak{S}_n$ denotes the symmetric group which acts by permutation on the product $S \times \cdots \times S$. We have a canonical map, called the *Douady–Barlet map*

$$\begin{aligned} \rho \;:\; S^{[n]} &\longrightarrow S^{(n)}\,, \\ Z &\longmapsto \sum_{z \in Z} (\dim_{\mathbb{C}} \mathcal{O}_{Z,z}) z\,. \end{aligned} \tag{1.1.8}$$

Originally, J. Fogarty [**Fog86**] showed that $S^{[n]}$ is smooth for every smooth surface S and that the Douady–Barlet map is a resolution of singularities.

If S is projective, then by GAGA the Douady space is nothing but the *Hilbert scheme* and the Douady–Barlet map is then called *Hilbert–Chow morphism.*

1.1.9. Generalized Kummer manifolds. Another example is also due to [**Bea84**]. We start with an complex two–torus S. The Douady space of points $S^{[n+1]}$ is a holomorphic symplectic manifold but not simply connected. Consider the composition

$$S^{[n+1]} \longrightarrow S^{(n+1)} \longrightarrow S$$

where the first map is the Douady–Barlet map and the second is the usual summation in the abelian surface S. Let $S^{[[n]]} = K_n(S)$ denote the fiber of this morphism over $0 \in S$. Then A. Beauville [**Bea84**] showed that $S^{[[n]]}$ is an irreducible holomoprhic symplectic manifold of dimension $2n$, called *generalized Kummer manifold.* Note that for $n = 1$, we get the usual Kummer surface.

1.1.10. Moduli spaces of sheaves. Let S be a projective K3 surface. For a suitable primitive *Mukai vector* v in the even integral cohomology and a generic choice of a polarization H i.e. an ample line bundle, see Definition 4.2.16, every H–semistable sheaf with Mukai vector v is stable. Therefore the moduli space $M_H(v)$ of (semi)stable sheaves with Mukai vector v is smooth and an irreducible holomorphic symplectic manifold, see Theorem 4.2.20. Originally, S. Mukai [**Muk84**] noticed that moduli spaces of stable sheaves admit a holomorphic symplectic form. We deal with the theory of (semi)stable sheaves in Chapter 4. For the specific case of a holomorphic symplectic surface, see section 4.2.7. Note that a similar construction is possible for an abelian surface, but one has again to consider a fiber $K_H(v) \subset M_H(v)$ of a certain map, similar to 1.1.9, to obtain an irreducible holomorphic symlectic manifold, see Theorem 4.2.22.

The obtained examples are deformation equivalent to the examples in 1.1.7 or 1.1.9, cf. 1.1.12.

1.1.11. O'Grady's examples. K. O'Grady found exceptional examples in dimension six and ten. Similarly to 1.1.10, one considers moduli spaces $M_H(v)$ of semistable sheaves with H chosen generically. Here the Mukai vector v is chosen non primitive i.e. $M_H(v)$ admits honest semistable sheaves which corresponds to singular points in $M_H(v)$. The O'Grady manifolds are obtained by resolving $M_H(v)$, cf. [**O'G99**] for the resulting 10–dimensional example coming from a K3 and [**O'G03**] for the 6–dimensional example coming from an abelian surface. The resulting Betti numbers are $b_2 = 24$ and $b_2 = 8$, respectively. We will dwell later on these examples in section 4.5.

1.1.12. Deformations. Let $\pi : \mathcal{X} \to S$ be a family of complex manifolds i.e. π is a proper and flat holomorphic map between connected complex spaces $\mathcal{X}$ and S and for each t the associated fiber $\mathcal{X}_t \coloneqq \pi^{-1}(t)$ is a complex manifold. Denote by $o \in S$ a reference point.

Theorem 1.1.13 ([**Bea84**, Prop. 9, Rem. 10], [**GHJ03**, Prop. 22.2]) *Assume $\mathcal{X}_o$ to be irreducible holomorphic symplectic. Let $t \in S$ such that $\mathcal{X}_t$ is a Kähler manifold, then $\mathcal{X}_t$ is irreducible holomorphic symplectic.*

One therefore introduces the following notion. An irreducible holomorphic symplectic manifold is called of $K3^{[n]}$*–type* or $K3^{[n]}$*–type manifold* if it is deformation equivalent to $S^{[n]}$ for a K3 surface S. Similarly ones speaks of irreducible holomorphic symplectic manifolds of *generalized Kummer type*, *O'Grady* 10*–type* and *O'Grady* 6*–type*.

Remark 1.1.14 Up to deformation, *all known* irreducible holomorphic symplectic manifolds are given by the examples above.

1.1.15. Hyperkähler versus symplectic. Irreducible holomorphic symplectic manifolds are also called *(compact) hyperkähler manifolds*, we shortly explain why.

Recall that an *almost complex structure* on a smooth manifold M is a smooth endomorphism $J : TM \to TM$ of the tangent bundle i.e. a smooth $(1,1)$–tensorfield such that $J^2 = -\mathrm{id}$.

Definition 1.1.16 A *hyperkähler structure* on a Riemannian manifold (M, g) consists of three almost complex structures I, J and K such that

(i) they are compatible with the metric g i.e.

$$g(I\cdot, I\cdot) \;=\; g(J\cdot, J\cdot) \;=\; g(K\cdot, K\cdot) \;=\; g(\cdot,\cdot)\,,$$

(ii) they satisfy the quaternionic relation i.e. $IJ = K$,

(iii) they are parallel with respect to the Levi–Civita connection ∇ of g i.e.

$$\nabla I \;=\; \nabla J \;=\; \nabla K \;=\; 0\,.$$

The tuple (M, g, I, J, K) is called *hyperkähler manifold.*

Remark 1.1.17 Let (M, I, J, K) be a hyperkähler manifold.

(i) For a point $x \in M$ the tangent space T_xM is a quaternionic vector space with respect to I_x, J_x and K_x. Therefore the real dimension of M is $\dim_{\mathbb{R}} M = 4n$ a multiple of four.

(ii) Parallelity of the almost complex structures with respect to the Levi–Civita connection imply that each tuple (M, I, g), (M, J, g) and (M, K, g) is a Kähler manifold. In particular each of the almost complex structures is integrable i.e. is a complex structure.

(iii) Choose a point $(p_1, p_2, p_3) \in \mathbb{S}^2$ in the unit two–sphere. Then $p_1I + p_2J + p_3K$ is again a complex structure by linearity of the Levi–Civita connection. Therefore a hyperkähler structure comes with a whole sphere of complex structures.

As usual for Riemannian manifolds, there is a holonomy characterization. With the holonomy principle one gets the following.

Theorem 1.1.18 [**Bau08**, Satz 5.24] *A Riemannian manifold (M, g) admits a hyperkähler structure if and only if its holonomy group* $\mathrm{Hol}(M, g)$ *is contained in the (compact) symplectic group* $\mathrm{Sp}(n)$*.*

Further, an easy computation shows the following.

Proposition 1.1.19 *Let (M, g, I, J, K) be a hyperkähler manifold. Then the two–forms $\omega_I \coloneqq g(I\cdot, \cdot)$, $\omega_J \coloneqq g(J\cdot, \cdot)$ and $\omega_K \coloneqq g(K\cdot, \cdot)$ are holomorphic symplectic on the complex manifolds (M, I), (M, J) and (M, K) respectively.*

The canonical question is, when they are indeed irreducible holomorphic symplectic.

Theorem 1.1.20 ([**Bea84**, Prop. 2], [**GHJ03**, Prop. 23.3]) *Let (M, g, I, J, K) be a compact hyperkähler manifold with* $\mathrm{Hol}(M, g) = \mathrm{Sp}(n)$*. Then (M, I), (M, J) and (M, K) are irreducible holomorphic symplectic manifolds.*

Conversely, by Yau's solution of the Calabi conjecture, cf. [**GHJ03**, 5.I], one has the following.

Theorem 1.1.21 ([**Bea84**, Prop. 3, Prop. 4], [**GHJ03**, Thm. 23.5]) *Let X be an irreducible holomorphic symplectic manifold and α a Kähler form on X. Then there exists precisely one Kähler metric g with associated Kähler form ω sucht that*

$$[\omega] = [\alpha] \;\; \textit{in} \;\; H^2(X, \mathbb{R}) \quad \textit{and} \quad \mathrm{Hol}(X, g) = \mathrm{Sp}(n)\,.$$

We conclude that the notion of irreducible holomorphic symplectic and compact hyperkähler manifolds with holonomy $\mathrm{Sp}(n)$ are essentially the same.

Although we do not use the hyperkähler structure in this thesis, the relation between holomorphic symplectic and differential geometry reflects the fact how extensive the whole theory is. For instance, M. Verbitsky [**Ver13**] used the hyperkähler structure for his Global Torelli Theorem, see Theorem 1.3.8.

1.2. Beauville–Bogomolov–Fujiki quadratic form

On every compact complex surface we have the well known intersection pairing on the integral second cohomology. On irreducible holomorphic symplectic manifolds there is a natural generalization, called the *Beauville–Bogomolov–Fujiki quadratic form*. The second integral cohomology together with the associated bilinear form defines a lattice in sense of Appendix A. We also refer to this section for basic notation and definitions from lattice theory.

Let X be an irreducible holomorphic symplectic manifold of dimension $2n$. We fix a *normalized* holomorphic symplectic form $\sigma \in H^0(X, \Omega_X^2)$ i.e. $\int_X \sigma\bar{\sigma} = 1$. Then we define the quadratic form $q'_X : H^2(X, \mathbb{C}) \to \mathbb{C}$

$$q'_X(\alpha) \;\coloneqq\; \frac{n}{2}\int_X \alpha^2(\sigma\bar{\sigma})^{n-1} + (1-n)\left(\int_X \alpha\sigma^{n-1}\bar{\sigma}^n\right)\left(\int_X \alpha\sigma^n\bar{\sigma}^{n-1}\right) \tag{1.2.1}$$

Clearly, for real $\alpha \in H^2(X, \mathbb{R})$ also $q'_X(\alpha)$ is real. The natural question is, when the quadratic form restricts to an integral form on $H^2(X, \mathbb{Z})$.

Theorem 1.2.2 ([**Bea84**, Thm. 5], [**Fuj87**], [**GHJ03**, Prop. 23.11, 23.14]) *There is a positive number $c \in \mathbb{R}$ such that*

$$q'(\alpha)^n \;=\; c\int_X \alpha^{2n}$$

for all $\alpha \in H^2(X, \mathbb{C})$. Therefore, q'_X can be renormalized to a form q_X which is a primitive and integral quadratic form on $H^2(X, \mathbb{Z})$. The signature of the form is $(3, b_2(X) - 3)$. The holomorphic symplectic form σ satisfies

$$q_X(\sigma) \;=\; 0 \quad \textit{and} \quad q_X(\sigma + \bar{\sigma}) > 0\,.$$

Definition 1.2.3 Let X be an irreducible holomorphic symplectic manifold.

(i) The renormalized form $q_X : H^2(X, \mathbb{Z}) \to \mathbb{Z}$ of Theorem 1.2.2 is called *Beauville–Bogomolov* or more precisely *Beauville–Bogomolov–Fujiki* quadratic form. Throughout the text, the associated nondegenerate bilinear form to q_X is denoted by $(\cdot, \cdot)$. Therefore $H^2(X, \mathbb{Z})$ with $(\cdot, \cdot)$ is a lattice in sense of Appendix A. Note that the last statement of the Theorem means, that $(\sigma, \sigma) = 0$ and $(\sigma, \bar{\sigma}) > 0$.

(ii) If one writes

$$c_X \frac{(2n)!}{n!2^n} q_X(\alpha) \;=\; \int_X \alpha^{2n}$$

for $\alpha \in H^2(X,\mathbb{Z})$, then the positive rational number $c_X \in \mathbb{Q}$ is called *Fujiki constant* which is invariant under deformation of the manifold X by [**Fuj87**].

1.2.4. The K3$^{[n]}$–type lattice. For the Douady space $S^{[n]}$ of a K3 surface S, A. Beauville [**Bea84**, Prop. 6] computed for $n \geq 2$, that there is a canonical isomorphism

$$H^2(S^{[n]},\mathbb{Z}) \;\cong\; H^2(S,\mathbb{Z}) \oplus \langle 2-2n\rangle\,.$$

It is well known that $H^2(S,\mathbb{Z})$ of an K3 surface is isometric to (abstract) K3 lattice Λ_{K3}, see (A.0.4), which is of signature $(3,19)$. Therefore $H^2(X,\mathbb{Z})$ for every K3$^{[n]}$–type manifold is isometric fo the *(abstract) K3$^{[n]}$–lattice*

$$\Lambda_{\mathrm{K3}} \oplus \langle 2-2n\rangle = E_8(-1)^{\oplus 2} \oplus \mathrm{U}^{\oplus 3} \oplus \langle 2-2n\rangle\;, \tag{1.2.5}$$

which is of signature $(3,20)$.

1.2.6. The generalized Kummer lattice. Similarly as for K3 surfaces, if we have an complex two torus S, A. Beauville [**Bea84**, Prop. 8] showed for $n \geq 2$, that there is a canonical isomorphism

$$H^2(S^{[n]},\mathbb{Z}) \;\cong\; H^2(S,\mathbb{Z}) \oplus \langle -(2+2n)\rangle\,.$$

The cohomology $H^2(S,\mathbb{Z})$ of a two torus is isometric to $\mathrm{U}^{\oplus 3}$. Therefore $H^2(X,\mathbb{Z})$ for a generalized Kummer manifold X is isometric to the *(abstract) generalized Kummer lattice*

$$\mathrm{U}^{\oplus 3} \oplus \langle -(2+2n)\rangle\;, \tag{1.2.7}$$

which is of signature $(3,4)$.

1.3. Marked pairs, moduli and Torelli

Recall the following basic notions, definitions and statements from deformation theory applied to an irreducible holomorphic symplectic manifold X, cf. [**Kod86**].

- Since $H^0(X,\mathcal{T}_X) \cong H^0(X,\Omega^1_X) = 0$, cf. Remarks 1.1.2 and 1.1.4, an universal family $\pi : \mathfrak{X} \to \mathrm{Def}(X)$ exists, which is known as the *Kuranishi family*.
- By F. Bogomolov [**Bog78**], the deformation space $\mathrm{Def}(X)$ is unobstructed, that is $\mathrm{Def}(X)$ is smooth.
- Usually we denote the reference point by $o \in \mathrm{Def}(X)$ i.e. $\mathfrak{X}_o := \pi^{-1}(o) = X$. We will view the base space $\mathrm{Def}(X)$ sometimes as a germ but also as a representative which we usually choose small enough i.e. simply connected and that all fibers are irreducible holomorphic symplectic, cf. 1.1.12. In this case, we also denote the fibers by $\mathfrak{X}_t := \pi^{-1}(t)$ for $t \in \mathrm{Def}(X)$.
- There is always an isomorphism $\mathcal{T}_o\,\mathrm{Def}(X) \cong H^1(X,\mathcal{T}_X)$. We have

$$h^1(X,\mathcal{T}_X) = h^{1,1}(X) = b_2(X) - 2$$

and since $\mathrm{Def}(X)$ is smooth, the dimension of it is

$$\dim \mathrm{Def}(X) = \dim \mathcal{T}_o \mathrm{Def}(X) = b_2(X) - 2\,.$$

For this section, we fix an irreducible holomorphic symplectic manifold X_0 and set $\Lambda \coloneqq H^2(X_0, \mathbb{Z})$. Further, let X be an irreducible holomorphic symplectic manifold deformation equivalent to X_0.

Definition 1.3.1 A *marking* or more precisely a Λ–*marking* on X is a choice of an isometry $\eta : H^2(X, \mathbb{Z}) \to \Lambda$. The tuple (X, η) is then called a *marked pair*, more precisely a Λ–*marked pair* or a *marked irreducible holomorphic symplectic manifold.* Two marked pairs (X_i, η_i), $i = 1, 2$, are called *isomorphic* if there is a biholomorphic map $f : X_1 \to X_2$ such that $\eta_2 = \eta_1 \circ f^\star$.

1.3.2. Period map and local Torelli. Consider the Kuranishi family $\pi : \mathfrak{X} \to \mathrm{Def}(X)$ with $\mathfrak{X}_o \coloneqq \pi^{-1}(o) = X$ and choose a marking η_o on X. Then by Ehresmann's theorem [**Voi02**, Thm. 9.3] we can choose a trivialization $\Sigma : R^2\pi_\star\mathbb{Z} \to \Lambda$ extending the marking η_o i.e. $\Sigma_o = \eta_o$. One also calls Σ a *marking* of the family π.

Then define the *local period map* by

$$\begin{aligned} \mathcal{P} : \mathrm{Def}(X) &\longrightarrow \mathbb{P}(\Lambda_\mathbb{C})\,, \\ t &\longmapsto [\Sigma_t(H^{2,0}(\mathfrak{X}_t))] \end{aligned} \tag{1.3.3}$$

where $\Lambda_\mathbb{C} \coloneqq \Lambda \otimes \mathbb{C}$ and $\Sigma_t : (R^2\pi_\star\mathbb{Z})_t \otimes_\mathbb{Z} \mathbb{C} \to \Lambda_\mathbb{C}$ is the induced map of stalks. By Theorem 1.2.2, it takes values in the *period domain of type* Λ (cf. [**GHJ03**, 22.3, 25.2]), namely

$$\Omega_\Lambda \ \coloneqq\ \{p \in \mathbb{P}(\Lambda_\mathbb{C}) \mid (p,p) = 0 \text{ and } (p, \bar{p}) > 0\}$$

which is connected since the signature of $(\cdot,\cdot)$ is $(3, \mathrm{rk}\,\Lambda - 3)$, see [**Bea84**, Thm. 5].

Theorem 1.3.4 (Local Torelli, [**Bea84**, Thm. 5]) *If* $\mathrm{Def}(X)$ *is chosen small enough, the period map* $\mathcal{P} : \mathrm{Def}(X) \to \Omega_\Lambda$ *is an open embedding.*

1.3.5. Moduli of marked pairs and Global Torelli. Fix a lattice Λ which is isometric to the second cohomology of an irreducible holomorphic symplectic manifold. The local Torelli ensures the construction of the following moduli space. There exists a *moduli space of marked pairs*

$$\mathfrak{M}_\Lambda \ \coloneqq\ \{(X,\eta)\ \Lambda\text{–marked pair }\}\,/ \cong \tag{1.3.6}$$

where $\cong$ denotes the relation of Definition 1.3.1. The complex structure can be constructed by gluing all deformation spaces $\mathrm{Def}(X)$ of irreducible holomorphic symplectic manifolds with $H^2(X, \mathbb{Z})$ isometric to Λ. This gives a non–Hausdorff complex manifold of dimension $\mathrm{rk}\,\Lambda - 2$, cf. [**Huy12**, Prop. 4.3]. In particular for given $(X, \eta) \in \mathrm{Def}(X)$, there is a holomorphic map $\mathrm{Def}(X) \hookrightarrow \mathfrak{M}_\Lambda$ which identifies $\mathrm{Def}(X)$ and a neighborhood of (X, η) biholomorphically.

The *global period map* is

$$\mathcal{P} \;:\; \mathfrak{M}_\Lambda \longrightarrow \Omega_\Lambda\,, \qquad (X,\eta) \longmapsto [\eta(H^{2,0}(X))] \tag{1.3.7}$$

and is a local biholomorphism by the Local Torelli. If one takes an arbitrary connected component $\mathfrak{M}_\Lambda^\circ$ of $\mathfrak{M}_\Lambda$ then by a result of D. Huybrechts [**GHJ03**, Prop. 25.12] the restriction $\mathcal{P} : \mathfrak{M}_\Lambda^\circ \to \Omega_\Lambda$ is surjective.

Recall that two points x and y in a topological space are called *inseparable* if every open neighborhoods U of x and V of y have nonempty intersection $U \cap V \neq \emptyset$.

The Torelli for K3 surfaces states that two K3 surfaces are biholomorphic if and only if there is an isometry of Hodge structures between the corresponding integral second cohomologies. We state the celebrated Global Torelli for hyperkähler manifolds which is a generalization and mainly due to M. Verbitsky [**Ver13**] and D. Huybrechts. For an overview, see [**Mar11**, 2.] and [**Huy12**].

Theorem 1.3.8 (Global Torelli) *Let $\mathcal{P}_\circ$ denote the restriction of the global period map to a fixed connected component $\mathfrak{M}_\Lambda^\circ$ of the moduli of marked pairs $\mathfrak{M}_\Lambda$. Then the following statements hold.*

(*i*) [**Ver13**, Thm. 1.16] *The fiber $\mathcal{P}_\circ^{-1}(p)$ consists of pairwise inseparable points for all $p \in \Omega_\Lambda$.*

(*ii*) [**Huy12**, Prop. 4.7] *If (X_1, η_1) and (X_2, η_2) are two inseparable points of $\mathfrak{M}_\Lambda$, then X_1 and X_2 are bimeromorphic.*

(*iii*) [**Ver13**, Thm. 4.24], [**Huy12**] *If (X_1, η_1) and (X_2, η_2) are two points of $\mathfrak{M}_\Lambda^\circ$ with $\mathcal{P}(X_1, \eta_2) = \mathcal{P}(X_2, \eta_2)$, then (X_1, η_1) and (X_2, η_2) are inseparable points of $\mathfrak{M}_\Lambda^\circ$.*

See also the Hodge theoretic Global Torelli 1.5.14.

1.3.9. Deformation of pairs. We are interested in the deformation space of a pair (X, L) where X is irreducible holomorphic symplectic and L is a line bundle.

Definition 1.3.10 [**Mar13**, 5.2] Let X_i, $i = 1, 2$, denote two irreducible holomorphic symplectic manifolds, L_i holomorphic line bundles on X_i and e_i classes in $H^2(X_i, \mathbb{Z})$.

(*i*) The pairs (X_1, e_1) and (X_2, e_2) are called *deformation equivalent* if there exists a familty $\pi : \mathcal{X} \to S$ of irreducible holomorphic symplectic manifolds over a connected complex space S with finitely many irreducible components, a section e of $R^2\pi_\star\mathbb{Z}$, points t_i in S such that $\mathcal{X}_{t_i} = X_i$ and $e_{t_i} = e_i$.

(*ii*) The pairs (X_1, L_1) and (X_2, L_2) are called *deformation equivalent* if there exists a family $\pi : \mathcal{X} \to S$ of irreducible holomorphic symplectic manifolds over a connected complex space S with finitely many irreducible components, a line bundle $\mathcal{L}$ on $\mathcal{X}$, points t_i in S such that $\mathcal{X}_{t_i} = X_i$ and $\mathcal{L}_{\mathcal{X}_{t_i}} = L_i$.

Let $L \in \mathrm{Pic}(X) \cong \mathrm{NS}(X)$ (cf. Remark 1.1.4) denote a line bundle on an irreducible holomorphic symplectic manifold X. Choose a Λ–marking η on X, set

$u \coloneqq \eta(c_1(L))$ and consider the hyperplane section

$$\Omega_{u^\perp} \ \coloneqq \ \Omega_\Lambda \cap u^\perp \ = \ \{p \in \Omega_\Lambda \mid (p,u) = 0\} \tag{1.3.11}$$

If $\mathcal{P} : \mathrm{Def}(X) \to \Omega_\Lambda$ denotes the period map we can define a smooth hypersurface germ

$$\mathrm{Def}(X,L) \ \coloneqq \ \mathcal{P}^{-1}(\Omega_{u^\perp}),$$

cf. [**GHJ03**, 26.1]. Set $\mathfrak{X}_L \coloneqq \mathcal{P}^{-1}(\mathrm{Def}(X,L))$. Then by abuse of notation, we denote the restriction $\pi : \mathfrak{X}_L \to \mathrm{Def}(X,L)$ of the Kuranishi family also by π.

Proposition 1.3.12 [**Bea84**, Cor. 1] *There exists a unique line bundle $\mathcal{L}$ on $\mathfrak{X}_L$ such that $\mathcal{L}|_X = L$. The family $\pi : \mathfrak{X}_L \to \mathrm{Def}(X,L)$ together with this line bundle L is universal in the following sense. Every deformation $(\mathfrak{X}_S \to S, \mathcal{G})$ with $\mathcal{G} \in \mathrm{Pic}(\mathfrak{X}_S)$ of the pair (X,L) in sense of* Definition 1.3.10 (ii) *is isomorphic to the pullback of $(\mathfrak{X}_L \to \mathrm{Def}(X,L), \mathcal{L})$ via a uniquely determined map $S \to \mathrm{Def}(X,L)$.*

We refer to this universal family $\pi : \mathfrak{X}_L \to \mathrm{Def}(X,L)$ as the *Kuranishi family* of the pair (X,L).

Remark 1.3.13 Note that we can reformulate (ii) of Definition 1.3.10 as the following.

- The pairs (X_1,L_1) and (X_2,L_2) are called *deformation equivalent* if there exists a family $\pi : \mathcal{X} \to S$ of irreducible holomorphic symplectic manifolds over a connected complex space S with finitely many irreducible components, a section e of $R^2\pi_\star\mathbb{Z}$ which is everywhere of Hodge type $(1,1)$, points t_i in S such that $\mathcal{X}_{t_i} = X_i$ and $e_{t_i} = c_1(L_i)$.

Clearly, $e_t \coloneqq c_1(\mathcal{L}_t)$ would give such a section. Conversely, given a section e as in the alternative definition, we get a line bundle L_t on $\mathcal{X}_t$ corresponding to $e_t \in H^{1,1}(\mathcal{X}_t,\mathbb{Z})$ with respect to the isomorphism $\mathrm{Pic}(\mathcal{X}_t) \cong H^{1,1}(\mathcal{X}_t,\mathbb{Z})$ since $\mathcal{X}_t$ is irreducible holomorphic symplectic. Then the Kuranishi family of the pair $(\mathcal{X}_t, L_t)$ gives an universal line bundle on the respective total space for every $t \in S$. Those line bundles glue to a line bundle $\mathcal{L}$ on $\mathcal{X}$ with the property $c_1(\mathcal{L}_t) = e_t$.

1.4. Orientation

We summarize section 4. of [**Mar11**]. Let $b_2 > 0$ a positive integer and Λ be an even lattice of signature $(3, b_2 - 3)$. Define

$$\widetilde{\mathcal{C}}_\Lambda \ \coloneqq \ \{x \in \Lambda_\mathbb{R} \mid (x,x) > 0\}\ .$$

We have the following.

Lemma 1.4.1 [**Mar11**, Lem. 4.1] *If $W \subset \Lambda_\mathbb{R}$ is a three dimensional subspace such that the bilinear form of Λ is positive definite on it, then $W \setminus \{0\}$ is a deformation retract of $\widetilde{\mathcal{C}}_\Lambda$. Therefore $H^2(\widetilde{\mathcal{C}}_\Lambda,\mathbb{Z}) \cong \mathbb{Z}$ is a free abelian group of rank one. The reflection R_u for $u \in \Lambda$ with $(u,u) \neq 0$ given by*

$$R_u(x) \ \coloneqq \ (x,x) - 2\frac{(u,x)}{(u,u)}u\,,$$

acts on $H^2(\widetilde{\mathcal{C}}_\Lambda, \mathbb{Z})$

- *as* $+1$ *if* $(e,e) < 0$ *and*
- *as* -1 *if* $(e,e) > 0$,

therefore it defines a generator of $H^2(\widetilde{\mathcal{C}}_\Lambda, \mathbb{Z})$.

In particular, the Lemma implies that $\widetilde{\mathcal{C}}_\Lambda$ is connected, as $H_0(\widetilde{\mathcal{C}}_\Lambda, \mathbb{Z}) = H_0(W \setminus \{0\}, \mathbb{Z}) = \mathbb{Z}$.

Definition 1.4.2 An *orientation of* $\widetilde{\mathcal{C}}_\Lambda$ is a choice of a generator of $H^2(\widetilde{\mathcal{C}}_\Lambda, \mathbb{Z}) \cong \mathbb{Z}$.

By speaking of oriented isometries of the lattice Λ, we mean isometries which preserve the orientation of $\widetilde{\mathcal{C}}_\Lambda$ in sense of the definition above: every isometry $g : \Lambda \to \Lambda$ induces a homeomorphism $g : \widetilde{\mathcal{C}}_\Lambda \to \widetilde{\mathcal{C}}_\Lambda$, therefore we have a morphism

$$\begin{aligned} \mathrm{O}(\Lambda) &\longrightarrow \mathrm{Aut}(H^2(\widetilde{\mathcal{C}}_\Lambda, \mathbb{Z})) \cong \{\pm 1\} \\ g &\longmapsto g^\star \,. \end{aligned} \tag{1.4.3}$$

Definition 1.4.4 The morphism in (1.4.3) above is also called *spinor norm*. Its kernel is denoted by $\mathrm{O}^+(\Lambda)$ and isometries in it are called *orientation preserving*.

For a primitive element $u \in \Lambda$ with $(u,u) > 0$ we can consider the hyperplane section

$$\Omega_{u^\perp} \;:=\; \Omega_\Lambda \cap u^\perp \;=\; \{p \in \Omega_\Lambda \mid (p,u) = 0\} \tag{1.4.5}$$

in the period domain. Since $(u,u) > 0$, the signature of $u^\perp \subset \Lambda_\mathbb{R}$ is $(2, b_2 - 3)$, hence $\Omega_{u^\perp}$ has two connected components.

If $p = \mathbb{C} \cdot \sigma \in \Omega_{u^\perp}$ is a period orthogonal to u, then the bilinear form of Λ restricted to the subvector space

$$W_p \;:=\; \mathrm{Re}(p) \oplus \mathrm{Im}(p) \oplus \mathbb{R} \cdot u \tag{1.4.6}$$

is positive definite. Note that the conditions $(p,p) = 0$ and $(p,\bar{p}) > 0$ are crucial for the bilinear form being positive definite on W_p: from the first and the second we get

$$(\mathrm{Re}(\sigma), \mathrm{Re}(\sigma)) = (\mathrm{Im}(\sigma), \mathrm{Im}(\sigma)) \text{ and } (\mathrm{Re}(\sigma), \mathrm{Re}(\sigma)) + (\mathrm{Im}(\sigma), \mathrm{Im}(\sigma)) > 0\,, \tag{1.4.7}$$

respectively. The subvector space W_p has the canonical ordered basis

$$(\mathrm{Re}(\sigma), \mathrm{Im}(\sigma), u)\,, \tag{1.4.8}$$

which defines an orientation in the ordinary sense i.e. a volume form $\beta(\sigma) := \mathrm{Re}(\sigma)^\star \wedge \mathrm{Im}(\sigma)^\star \wedge u^\star$ of the manifold $W_p \setminus \{0\}$. The orientation $\beta(\sigma)$ does not depend on the choice of σ, indeed we have $\beta(\lambda\sigma) = |\lambda|\beta(\sigma)$ for any $\lambda \in \mathbb{C}$. Take the two sphere $\mathbb{S}^2 \subset W_p \setminus \{0\}$ in W_p. It is well known, that the basis (1.4.8) gives a volume form on $\mathbb{S}^2$ by restricting the two form

$$x_1 \,\mathrm{im}(\sigma)^\star \wedge u^\star + x_2 u^\star \wedge \mathrm{Re}(\sigma)^\star + x_3 \,\mathrm{Re}(\sigma)^\star \wedge \mathrm{im}(\sigma)^\star$$

to $\mathbb{S}^2$, where x_1, x_2, x_3 are the standard coordinates with respect to the basis (1.4.8). Use

$$H^2(\mathbb{S}^2, \mathbb{Z}) = H^2(W_p \setminus \{0\}, \mathbb{Z}) = H^2(\widetilde{\mathcal{C}}_\Lambda, \mathbb{Z}) \tag{1.4.9}$$

to obtain a generator of $H^2(\widetilde{\mathcal{C}}_\Lambda, \mathbb{Z})$ i.e. an orientation in sense of Definition 1.4.2. Obviously we end up with the other generator, if we change the orientation of W_p given by the basis (1.4.8).

Principle 1.4.10 *Let $u \in \Lambda$ be an element with $(u,u) > 0$. Any period $p \in \Omega_{u^\perp}$ orthogonal to u determines a generator of $H^2(\widetilde{\mathcal{C}}_\Lambda, \mathbb{Z}) \cong \mathbb{Z}$ i.e. an orientation of $\widetilde{\mathcal{C}}_\Lambda$. The two generators are distinguished by the two connected components of $\Omega_{u^\perp}$. Therefore a connected component of $\Omega_{u^\perp}$ determines an orientation of $\widetilde{\mathcal{C}}_\Lambda$*

For a period $p \in \Omega_\Lambda$ let $\Lambda(p)$ denote the integral Hodge structure of weight two of Λ determined by the period p, that is

$$\Lambda^{2,0}(p) = p\,, \quad \Lambda^{0,2}(p) = \bar{p} \quad \text{and} \quad \Lambda^{1,1}(p) = \{x \in \Lambda_\mathbb{C} \mid (x,p) = (x,\bar{p}) = 0\}\ . \tag{1.4.11}$$

As in the geometric situation, we also set

$$\Lambda^{1,1}(p,R) \ \coloneqq\ \{x \in \Lambda_R \mid (x,p) = 0\}$$

for $R \in \{\mathbb{Z}, \mathbb{R}\}$. Further consider the set

$$\mathcal{C}'_p \coloneqq \left\{x \in \Lambda^{1,1}(p,\mathbb{R}) \mid (x,x) > 0\right\}. \tag{1.4.12}$$

The restriction of the bilinear form to $\Lambda^{1,1}(p,\mathbb{Z})$ has signature $(1, b_2 - 3)$, see (1.4.7). Therefore $\mathcal{C}'_p$ has two connected components.

Let x be in $\mathcal{C}'_p$ with $p = \mathbb{C} \cdot \sigma$. Again we can define a subspace

$$W_x \ \coloneqq\ \mathrm{Re}(p) \oplus \mathrm{Im}(p) \oplus \mathbb{R} \cdot x \tag{1.4.13}$$

of $\Lambda_\mathbb{R}$ as in (1.4.6), such that the bilinear form is positive definite on it. Similarly as above, the ordered basis $(\mathrm{Re}(p), \mathrm{Im}(p), x)$ of $\Lambda_\mathbb{R}$ defines in the same way as in (1.4.9) a generator of $H^2(\widetilde{\mathcal{C}}_\Lambda, \mathbb{Z}) \cong \mathbb{Z}$ and again the generators are distinguished by the two connected components of $\mathcal{C}'_p$.

Principle 1.4.14 *An element x in $\mathcal{C}'_p$ for a period $p \in \Omega_\Lambda$ determines a generator of $H^2(\widetilde{\mathcal{C}}_\Lambda, \mathbb{Z}) \cong \mathbb{Z}$ i.e. an orientation of $\widetilde{\mathcal{C}}_\Lambda$. The two generators are distinguished by the two connected components of $\mathcal{C}'_p$. Therefore a connected component of $\mathcal{C}'_p$ determines an orientation of $\widetilde{\mathcal{C}}_\Lambda$.*

1.4.15. The geometric situation. Let $\mathfrak{M}_\Lambda$ denote the moduli space of isomorphism classes of marked pairs (X, η) of type Λ i.e. X is an irreducible holomorphic symplectic manifold and $\eta : H^2(X,\mathbb{Z}) \to \Lambda$ is a marking. Choose a connected component $\mathfrak{M}^\circ_\Lambda$ of $\mathfrak{M}_\Lambda$. Recall that for $(X,\eta) \in \mathfrak{M}^\circ_\Lambda$ there is a canonical choice for the connected component of

$$\mathcal{C}'_X \ \coloneqq\ \left\{x \in H^{1,1}(X,\mathbb{R}) \mid (x,x) > 0\right\}$$

namely the *positive cone* $\mathcal{C}_X$ which contains the Kähler cone $\mathcal{K}_X$ of X. Therefore, by Principle 1.4.14

$$\widetilde{\mathcal{C}}_X \;\coloneqq\; \widetilde{\mathcal{C}}_{H^2(X,\mathbb{Z})} \;=\; \{x \in H^2(X,\mathbb{R}) \mid (x,x) > 0\}$$

has a *natural orientation*, which determines an orientation in sense of Definition 1.4.2 of $\widetilde{\mathcal{C}}_\Lambda$ via the homeomorphism $\eta : \widetilde{\mathcal{C}}_X \cong \widetilde{\mathcal{C}}_\Lambda$.

Definition 1.4.16 We will refer to the orientation of $\widetilde{\mathcal{C}}_\Lambda$ (in sense of Definition 1.4.2) which is induced by the marking η and the natural orientation of $\widetilde{\mathcal{C}}_X$ for some (hence for all) marked pair (X,η) in $\mathfrak{M}^\circ_\Lambda$, as the orientation *compatible* to the connected component $\mathfrak{M}^\circ_\Lambda$ of the moduli of marked pairs.

Consider the period map

$$\mathcal{P} \;:\; \mathfrak{M}^\circ_\Lambda \longrightarrow \Omega_\Lambda\,, \quad (X,\eta) \longmapsto [\eta(H^{2,0}(X))]$$

and set $p \coloneqq \mathcal{P}(X,\eta)$. Then $\eta(H^{1,1}(X,\mathbb{R})) = \Lambda^{1,1}(p,\mathbb{R})$. An orientation of $\widetilde{\mathcal{C}}_\Lambda$ determines a connected component

$$\mathcal{C}_p \subset \mathcal{C}'_p \tag{1.4.17}$$

of $\mathcal{C}'_p$ by Principle 1.4.14. Equivalently, we can characterize the orientation compatible to $\mathfrak{M}^\circ_\Lambda$ by the condition $\eta(\mathcal{C}_X) = \mathcal{C}_p$ for all $(X,\eta) \in \mathfrak{M}^\circ_\Lambda$ with $p = \mathcal{P}(X,\eta)$.

Definition 1.4.18 If $u \in \Lambda$ is a class with $(u,u) > 0$, then let

$$\Omega^+_{u^\perp} \subset \Omega_{u^\perp}$$

denote the connected component of $\Omega_{u^\perp}$ which determines, cf. Principle 1.4.10, the orientation of $\widetilde{\mathcal{C}}_\Lambda$ which is compatible to $\mathfrak{M}^\circ_\Lambda$.

1.4.19. $\Omega^+_{\lambda^\perp}$ for an isotropic class. For the following see also [**Mar14**, 4.3]. We are still in the setting of 1.4.15, but now $\lambda \in \Lambda$ is a nontrivial isotropic class. We can still define a hyperplane section as in (1.4.5)

$$\Omega_{\lambda^\perp} \;\coloneqq\; \Omega_\Lambda \cap \lambda^\perp \;=\; \{p \in \Omega_\Lambda \mid (p,u) = 0\}\,. \tag{1.4.20}$$

Note that the bilinear form on $\lambda^\perp \subset \Lambda_\mathbb{R}$ is degenerate since λ is isotropic. The hyperplane section $\Omega_{\lambda^\perp}$ has two connected components and we can still obtain a natural connected component of it from the geometrical situation in the following way.

For $p \in \Omega_{\lambda^\perp}$, λ belongs to $\Lambda^{1,1}(p,\mathbb{R})$ and is contained in the boundary of one of the connected components of $\mathcal{C}'_p$ since λ is isotropic. For $(X,\eta) \in \mathfrak{M}^\circ_\Lambda$, either $\eta^{-1}(\lambda)$ or $\eta^{-1}(-\lambda)$ belongs to $\partial\mathcal{C}_X$. We assume that the former is the case, otherwise take $-\lambda$. Then consider only periods p in $\Omega_{\lambda^\perp}$ such that λ belongs to the closure of the distinguished connected component $\mathcal{C}_p$ in $\Lambda^{1,1}(p,\mathbb{R})$, see (1.4.17), determined by the orientation of $\widetilde{\mathcal{C}}_\Lambda$ compatible to $\mathfrak{M}^\circ_\Lambda$ i.e.

$$\Omega^+_{\lambda^\perp} \;\coloneqq\; \{p \in \Omega_{\lambda^\perp} \mid \lambda \in \partial\mathcal{C}_p\} \tag{1.4.21}$$

which is one of the connected components of $\Omega_{\lambda^\perp}$. Note that the only common element of the closures of the connected components of $\Omega_{\lambda^\perp}$ is the null vector, therefore $\Omega^+_{\lambda^\perp}$ of (1.4.21) is indeed one of the connected components of $\Omega_{\lambda^\perp}$. We refer to $\Omega^+_{\lambda^\perp}$ as the *compatible* connected component of $\Omega_{\lambda^\perp}$ with respect to the chosen connected component $\mathfrak{M}^\circ_\Lambda$ of the moduli of marked pairs.

1.5. Parallel transport and monodromy

Recall that a local system F on some connected and locally contractible ringed space X is a sheaf of A–modules locally given by the constant sheaf A for some commutative ring A. There are several equivalent ways to define the *parallel transport* $P_\gamma : F_p \to F_q$ in the local system F along a curve $\gamma : [0,1] \to X$ with $p := \gamma(0)$ and $q := \gamma(1)$. For an overview, see [**BF02**, 5.1] and [**Voi03**, 3.]

- The pullback $\gamma^{-1}F$ is a local system on the simply connected space $[0,1]$. By [**Voi03**, Prop. 3.9] it is a constant sheaf and we can compose the following isomorphisms

$$P_\gamma \;:\; F_p \longrightarrow \left(\gamma^{-1}F\right)([0,1]) \longrightarrow F_q$$

  which are the canonical stalk maps.
- Assume X to be a complex manifold. Then by [**Voi02**, 9.2.1] the holomorphic vector bundle $E := F \otimes_{\mathbb{Z}} \mathcal{O}_X$ comes with a canonical flat connection locally defined by $\nabla s := \sum_{j=1}^m df_j \otimes s_j$ with $s = \sum_{j=1}^m f_j \otimes s_j$ for a frame $s_1, \ldots, s_m$, called the *Gauss-Manin connection.* Then $E_p = F_p \otimes (\mathcal{O}_{X,p}/\mathfrak{m}_{X,p})$ and we have the ordinary parallel transport

$$P^\nabla_\gamma \;:\; E_p \longrightarrow E_q$$

  and can restrict it to F_p. Note that parallel transport of flat connections does only depend on the homotopy class of the curve, cf. [**KN63**, p. 93].
- If the local system $F = R^k\pi_\star A$ is a higher direct image of a proper holomorphic submersion $\pi : \mathcal{X} \to S$, then the parallel transport can be obtained with the fiber diffeomorphisms, [**Voi03**, 3.1.2]. Assume that $\pi : \mathcal{X} \to S$ is trivial over $U \subset S$ i.e. there is a diffeomorphism $\varphi_x : \pi^{-1}(U) = U \times \mathcal{X}_x$ for every $x \in U$. The fiber diffeomorphism is $\varphi_{x,y} := \varphi_y \circ \varphi_x^{-1}|_{\mathcal{X}_x} : \mathcal{X}_x \to \mathcal{X}_y$. By Künneth we have $R^k\pi_\star A|_U = H^k(\mathcal{X}_x, A)_U$. If $\gamma : [0,1] \to U$ is a way from p to q, then $P(\gamma) = (\varphi_{x,y}^{-1})^\star$ i.e.

$$(R^k\pi_\star A)_p = H^k(\mathcal{X}_p, A) \overset{(\varphi_{x,y}^{-1})^\star}{\longrightarrow} H^k(\mathcal{X}_q, A) = (R^k\pi_\star A)_q$$

By *monodromy* one means in general parallel transport along closed curves.

1.5.1. Monodromy for hyperkähler manifolds. In the geometry of hyperkähler manifolds, we are interested in *monodromy groups.*

Definition 1.5.2 Let X_i, $i = 1, 2$, denote two irreducible holomorphic symplectic manifolds.

(*i*) An isomorphism $P : H^2(X_1, \mathbb{Z}) \to H^2(X_2, \mathbb{Z})$ is called a *parallel transport operator* if there exists a family $\pi : \mathcal{X} \to S$ of irreducible holomorphic symplectic manifolds, points t_i such that $\mathcal{X}_{t_i} = X_i$ and a continuous path γ between t_1 and t_2 such that the parallel transport P_γ along γ in the local system $R^2\pi_\star\mathbb{Z}$ coincides with P.

(*ii*) In the case $X \coloneqq X_1 = X_2$, P is also called a *monodromy operator*.

(*iii*) The *monodromy group* is the subset $\mathrm{Mon}^2(X)$ of $\mathrm{Aut}(H^2(X, \mathbb{Z}))$ of all monodromy operators.

Remark 1.5.3 (*i*) Note that everything in the Definition above can be defined more general, by considering isomorphisms between the corresponding cohomology rings $H^\bullet(X_i, \mathbb{Z})$ with parallel transport in $R\pi_\star\mathbb{Z}$. But we do not need this general notion.

(*ii*) The monodromy group $\mathrm{Mon}^2(X)$ is indeed a subgroup of the automorphism group $\mathrm{Aut}(H^2(X, \mathbb{Z}))$, see [**Mar11**, Footnote 3].

(*iii*) If one views the monodromy in terms of the parallel transport of the Gauss–Manin connection ∇ in $R^2\pi_\star\mathbb{Z}$, then the group of monodromy operators obtained from a family $\pi : \mathcal{X} \to S$, as in the definition above, is nothing but the holonomy group $\mathrm{Hol}_p(\nabla)$, for a fixed point $p \in S$. It is well know that the holonomy group is a subgroup.

Let $\pi : \mathcal{X} \to S$ denote a deformation of the irreducible holomorphic symplectic manifold $X = \mathcal{X}_o$ and assume that π is trivial over a connected open set $U \subset S$ and denote by $\gamma_t : [0, 1] \to S$ a family curves from o to t. Then the family of parallel transports P_{γ_t} varies continuously in t (use the fiber diffeomorphisms). Therefore, when we consider for each $t \in U$ the Beauville–Bogomolov form $q_{\mathcal{X}_t}$ and take $v \in H^2(X, \mathbb{Z})$ then

$$U \ni t \longmapsto q_{\mathcal{X}_t}(P_{\gamma_t}(v)) \in \mathbb{Z}$$

is continuous, hence constant as U is connected. This means that the Beauville–Bogomolov forms Q_t on each fiber $\mathcal{X}_t$ fit together to a parallel section Q of

$$\mathrm{Sym}^2(R^2\pi_\star\mathbb{Z} \otimes \mathcal{O}_S)$$

i.e. $\nabla Q = 0$ for the Gauss–Manin connection. We have the following.

Lemma 1.5.4 *A parallel transport operator* $P : H^2(X_1, \mathbb{Z}) \to H^2(X_2, \mathbb{Z})$ *is an isometry of the lattices* $H^2(X_i, \mathbb{Z})$ *with respect to the Beauville–Bogomolov forms. In particular,* $\mathrm{Mon}^2(X)$ *is a subgroup of* $\mathrm{O}(H^2(X, \mathbb{Z}))$*.*

Fix an irreducible holomorphic symplectic manifold X_0 and set $\Lambda \coloneqq H^2(X_0, \mathbb{Z})$. By [**Mar11**, Lem. 7.5] the number of connected components $\mathfrak{M}^\circ_\Lambda$ of the moduli of marked pairs $\mathfrak{M}_\Lambda$ consisting of pairs (X, η) such that X is deformation equivalent to X is finite. Let τ denote the set of such connected components. We have a natural action of $\mathrm{O}(\Lambda)$ on τ defined by $g \cdot \mathfrak{M}^\circ_\Lambda \coloneqq \mathfrak{M}^\circ_\Lambda \circ g$ i.e. $g \cdot (X, \eta) \coloneqq (X, \eta \circ g)$. By [**Mar11**, Lem. 7.5] this action is transitive and each stabilizer is equal to the subgroup

$$\mathrm{Mon}^2(\mathfrak{M}^\circ_\Lambda) \;:=\; \eta \circ \mathrm{Mon}^2(X) \circ \eta^{-1} \subset \mathrm{O}(\Lambda) \quad \text{for } (X,\eta) \in \mathfrak{M}^\circ_\Lambda. \tag{1.5.5}$$

When $\mathrm{Mon}^2(X_0)$ is normal in $\mathrm{O}(H^2(X_0,\mathbb{Z}))$, then $\mathrm{Mon}^2(\mathfrak{M}^\circ_\Lambda) \subset \mathrm{O}(\Lambda)$ is equal for all $\mathfrak{M}^\circ_\Lambda \in \tau$. This is the case for $\mathrm{K3}^{[n]}$ and generalized Kummer manifolds, see the next subsection.

1.5.6. Monodromy results. We summarize the monodromy results for $\mathrm{K3}^{[n]}$ and generalized Kummer type manifolds of E. Markman and G. Mongardi, respectively.

Let Λ denote a non–degenerate lattice of signature $(3, b_2 - 3)$.

Definition 1.5.7 Let $\mathcal{W}(\Lambda)$ denote the subgroup of $\mathrm{O}^+(\Lambda)$ consisting of orientation preserving isometries acting as ± 1 on the discriminant $\Lambda^\vee/\Lambda$. Denote by

$$\chi : \mathcal{W}(\Lambda) \to \{\pm 1\}$$

the associated character. We also write $\mathcal{W}(X) := \mathcal{W}(H^2(X,\mathbb{Z}))$ for an irreducible holomorphic manifold X.

For a class $u \in \Lambda$ with $(u,u) \neq 0$ we have the rational reflection $R_u : \Lambda \to \Lambda$ defined by

$$R_u(x) \;:=\; x - 2\frac{(u,x)}{(u,u)}u\,. \tag{1.5.8}$$

If $(u,u) < 0$, then by Lemma 1.4.1 the reflection R_u is orientation preserving in sense of Definition 1.4.4 i.e. contained in $\mathrm{O}^+(\Lambda_\mathbb{Q})$.

Definition 1.5.9 Let Λ be a non–degenerate lattice of signature $(3, b_2 - 3)$. For a class $u \in \Lambda$ with $(u,u) \neq 0$, denote $\rho_u : \Lambda_\mathbb{Q} \to \Lambda_\mathbb{Q} \in \mathrm{O}^+(\Lambda_\mathbb{Q})$ the orientation preserving isometry defined by

$$\rho_u \;:=\; \begin{cases} R_u & \text{if } (u,u) < 0\,, \\ -R_u & \text{if } (u,u) > 0\,. \end{cases}$$

Remark 1.5.10 (*i*) If $(u,u) = \pm 2$, then R_u and ρ_u define honest integral isometries $\Lambda \to \Lambda$.

(*ii*) The action of R_u on $\Lambda^\vee$ for a $h \in \Lambda^\vee$ is

$$R_u(h)(x) \;=\; h(R_u(x)) \;=\; h(x) - (2\frac{f(u)}{(u,u)}u, x),$$

i.e. $R_u(h) = h \mod \Lambda$, hence for $(u,u) = \pm 2$ the isometry ρ_u is contained in $\mathcal{W}(\Lambda)$. More precisely we have

$$\chi(\rho_u) \;=\; \begin{cases} +1 & \text{if } (u,u) < 0\,, \\ -1 & \text{if } (u,u) > 0\,. \end{cases}$$

(*iii*) The isometry R_u satisfies $R_u(u) = -u$ and $R_u|_{u^\perp} = \mathrm{id}_{u^\perp}$, hence we have for the determinant $\det(R_u) = -1$. Therefore

$$\det(\rho_u) \;=\; \begin{cases} -1 & \text{if } (u,u) < 0\,, \\ (-1)^{b_2+1} & \text{if } (u,u) > 0\,. \end{cases}$$

Note that for the $\mathrm{K3}^{[n]}$ and generalized Kummer case b_2 is odd and for the O'Grady examples b_2 is even.

Theorem 1.5.11 (Markman, [**Mar11**, 9.]) *Let X be an $K3^{[n]}$–type manifold. Then* $\mathrm{Mon}^2(X)$ *is equal to the following sets.*

- [**Mar11**, Thm. 9.1] *The subgroup of* $\mathrm{O}^+(H^2(X,\mathbb{Z}))$ *generated by the isometries ρ_u for elements $u \in H^2(X,\mathbb{Z})$ with $(u,u) = \pm 2$.*
- [**Mar11**, Lem. 9.2] *The subgroup $\mathcal{W}(X)$ of* $\mathrm{O}^+(H^2(X,\mathbb{Z}))$*.*

There is also a third characterization we will come to later, where a monodromy invariant orbit of primitive isometric embeddings from the $\mathrm{K3}^{[n]}$ lattice into the Mukai lattice is used.

The generalized Kummer case is slightly different.

Theorem 1.5.12 (Mongardi, [**Mon14**, Thm. 2.3]) *Let X be a generalized Kummer n–type manifold. Then* $\mathrm{Mon}^2(X)$ *consists precisely of orientation preserving isometries $g \in \mathcal{W}(X)$ such that* $\chi(g) \cdot \det(g) = 1$*.*

In particular, for a generalized Kummer manifold X, $\mathrm{Mon}^2(X)$ is an index 2 sub group of $\mathcal{W}(X)$ as $|\mathcal{W}(X)/\mathrm{Mon}^2(X)| = |\operatorname{im}(\det\cdot\chi)| = 2$.

Corollary 1.5.13 *For a generalized Kummer type manifold X, the monodromy group* $\mathrm{Mon}^2(X)$ *is an index 2 sub group of $\mathcal{W}(X)$. The orientation preserving isometry $\rho_u \in \mathcal{W}(X)$ for a class $u \in H^2(X,\mathbb{Z})$ with $(u,u) = \pm 2$ defined in* Definition 1.5.9 *is never contained in* $\mathrm{Mon}^2(X)$*.*

Proof: The first statement we have just discussed. The second statement follows from Remark 1.5.10 (ii) and (iii). □

1.5.14. Hodge theoretic Global Torelli. We conclude the chapter by citing the following variant of Verbitsky's Global Torelli 1.3.8 which is due to E. Markman [**Mar11**, Thm. 1.3].

Theorem 1.5.15 [**Mar11**, Thm. 1.3] *Let X and Y be two irreducible holomorphic symplectic manifolds which are deformation equivalent.*

(*i*) *Then X and Y are bimeromorphic if and only if there exists a parallel transport operator $P : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ which is an isomorphism of integral Hodge structures.*

(*ii*) *Let* $P : H^2(Y, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ *be a parallel transport operator which is an isomorphism of integral Hodge structures. Then there exists a biholomorphism* $f : X \to Y$ *such that* $f^\star = P$ *if and only if* P *maps some Kähler class on* Y *to a Kähler class on* X*.*

CHAPTER 2

Lagrangian Fibrations

The chapter starts with an introduction to basic facts and statements about Lagrangian fibrations. In Section 2.2 the relation between Lagrangian fibrations and isotropic nef line bundles is explained. We conclude the Chapter with deformation and moduli theory of Lagrangian fibrations. The latter gives a geometrical interpretation of polarization types and monodromy invariants, which are introduced in the next chapters.

2.1. Basic facts

Definition 2.1.1 A n–dimensional complex subspace (or analytic subvariety) Y of a holomorpic symplectic manifold (X, σ) of dimension $2n$ is called *Lagrangian* if the restriction of the holomorphic symplectic form σ vanishes on the smooth part of Y.

Due to an observation of C. Voisin combined with the Kodaira's embedding theorem, Lagrangian submanifolds of an irreducible holomorphic symplectic manifold are always projective. A related statement is Proposition 3.4.4.

Proposition 2.1.2 [**Cam06**, Prop. 2.1] *Let Y be a Lagrangian submanifold of an irreducible holomorphic symplectic manifold X. Then Y is projective.*

Proof: Apply [**Cam06**, Prop. 2.1] to your favorite fiber Y of $X \times Y \to X$, $(x, y) \mapsto x$. Since $H^0(X, \Omega_X^2)$ is generated by the symplectic form, the restriction $r : H^0(X, \Omega_X^2) \to H^0(Y, \Omega_Y^2)$ is trivial. Then [**Cam06**, Prop. 2.1] states that Y must be projective. □

Due to D. Matsushita much is known about nontrivial fiber structures on irreducible holomorphic symplectic manifolds.

Theorem 2.1.3 (Matsushita, [**Mat99**], [**Mat00**], [**Mat01**], [**Mat03**]) *Let $f : X \to B$ be a surjective holomorphic map with connected fibers from an irreducible holomorphic symplectic manifold X of dimension $2n$ to a normal complex space B such that $0 < \dim B < 2n$ (that is, f is a fibration, see* Remark 2.1.5*). Then the following statements hold.*

(*i*) *B is projective of dimension n and its Picard number is $\rho(B) = 1$.*
(*ii*) *For all $t \in B$ the fiber $X_t := f^{-1}(t)$ is a Lagrangian subvariety.*
(*iii*) *If X_t is smooth then it is a complex torus i.e. an abelian variety by* Proposition 2.1.2.

Definition 2.1.4 (Lagrangian Fibration) Such a fibration $f : X \to B$ as in the Theorem is called a *Lagrangian fibration*.

- If X is of $\mathrm{K3}^{[n]}$–type, generalized Kummer type, O'Grady 10–type or 6–type, then the Lagrangian fibration f is called of *$K3^{[n]}$–type*, *generalized Kummer type*, *O'Grady* 10*–type* or *O'Grady* 6*–type* respectively.
- We also mean by a *$K3^{[n]}$–type fibration* a Lagrangian fibration of $\mathrm{K3}^{[n]}$–type and similarly with the other deformation types.

Sometimes, one says that a holomorphic map $f : X \to \mathbb{P}^N$ *defines* or *is a Lagrangian fibration* if $f : X \to \mathrm{im}(f) \subset \mathbb{P}^N$ is a Lagrangian fibration in the sense above.

Remark 2.1.5 (*i*) More generally a *fibration* on a complex space X is a proper surjective holomorphic map $f : X \to B$ with connected fibers with B a normal complex space such that $0 < \dim B < \dim X$. If X is a holomorphic symplectic manifold, then f is called a *Lagrangian fibration* if every irreducible component of every fiber is a Lagrangian subvariety. But we do not need this general notion and in this work we mean by a Lagrangian fibration always a Lagrangian fibration defined on an irreducible holomorphic symplectic manifold i.e. a map as in Theorem 2.1.3.

(*ii*) The statement in Matsushita's theorem that a smooth fiber Y of f has to be a complex torus can be proved by using the holomorphic analogue [**Mar88**, Prop. 1] of the classical Liouville theorem from real symplectic geometry [**Arn89**, 49 A]. Alternatively, one can argue that the Albanese map of Y is an isomorphism, cf. [**HO09**, Prop. 3.1].

If the base of the Lagrangian fibration is smooth even more is known due to a deep result of J.-M. Hwang which was recently slighty generalized by C. Lehn and D. Greb to the non–projective case.

Theorem 2.1.6 (Hwang, [**Hwa08**], [**GL14**]) *Let $f : X \to B$ be a Lagrangian fibration such that B is smooth and $\dim X = 2n$. Then $B \cong \mathbb{P}^n$.*

The general conjecture is that the base is always the projective space.

Conjecture 2.1.7 *Let $f : X \to B$ be a Lagrangian fibration. Then B is smooth i.e. B is a projective space.*

There are partial results concerning this conjecture for the $\mathrm{K3}^{[n]}$ and generalized Kummer type by E. Markman and K. Yoshioka respectively in combination with a result of D. Matsushita. Those results are corollaries of more general statements which we discuss later in section 2.2 about isotropic and nef line bundles.

Theorem 2.1.8 *Let $f : X \to B$ be a Lagrangian fibration with $\dim X = 2n$.*

(*i*) [**Mar11**, Thm. 1.3, Rem. 1.8] [**Mat13**, Thm. 1.2, Cor. 1.1] *If X is of $K3^{[n]}$–type, then $B \cong \mathbb{P}^n$.*

(*ii*) [**Yos12**, Appendix] [**Mat13**, Thm. 1.2, Cor. 1.1] *If X is of generalized Kummer type, then $B \cong \mathbb{P}^n$.*

We give the basic examples of Lagrangian fibrations.

2.1.9. Elliptic K3 surfaces and induced Lagrangian fibrations. In dimension two, a Lagrangian fibration is nothing but a genus one fibration $f : S \to \mathbb{P}^1$ on a K3 surface S. Sometimes f is called elliptic K3. It induces a higher dimensional example by taking the Douady space of n points of it and using the Douady–Barlet map $\rho : S^{[n]} \longrightarrow S^{(n)}$ (cf. 1.1.7), i.e.

$$S^{[n]} \xrightarrow{\rho} S^{(n)} \overset{f\times\cdots\times f}{\longrightarrow} (\mathbb{P}^1)^{(n)} \cong \mathbb{P}^n$$

is a Lagrangian fibration on the Douady space $S^{[n]}$, cf. 1.1.7, by Matsushita's Theorem 2.1.3. Note that the generic smooth fiber is the product of elliptic curves coming from the smooth fibers of $f : S \to \mathbb{P}^1$.

2.1.10. Lagrangian fibrations by elliptic two–tori. We start with an elliptic complex two–torus $p : S \to E$ i.e. p is surjective, S is a complex two–torus and E is an elliptic curve. We have the generalized Kummer manifold 1.1.9 $S^{[[n]]}$ and the following map

$$S^{[[n]]} \xrightarrow{\subset} S^{[n+1]} \xrightarrow{\rho} S^{(n+1)} \overset{p\times\cdots\times p}{\longrightarrow} E^{(n+1)} \cong \mathbb{P}^n \times E\,.$$

This map and the projection from $\mathbb{P}^n \times E$ to $\mathbb{P}^n$ defines a Lagrangian fibration $S^{[[n]]} \to \mathbb{P}^n$ by Matsushita's Theorem 2.1.3. Let F denote a smooth fiber of p, then the fiber of the Lagrangian fibration $S^{[[n]]} \to \mathbb{P}^n$ is isomorphic to the abelian subvariety of F^{n+1} given by the equation $x_1 + \ldots + x_{n+1} = 0$ for $(x_1, \ldots, x_{n+1}) \in F^{n+1}$.

2.1.11. Beauville–Mukai systems. Let $M_H(v)$ or $K_H(v)$ be a moduli space as in 1.1.10. If the Mukai vector is chosen in the form $v = (0, c_1(D), s)$ with D a big and nef divisor, then the support morphism, see Definition 4.4.7, $F \mapsto \mathrm{supp}(F)$ can be used to define Lagrangian fibrations $M_H(v) \to |D|$ or $K_H(v) \to |D|$, respectively. These Lagrangian fibration are known as *Beauville–Mukai systems*, which will be defined more precisely in section 4.4.

In a similar fashion one can get examples of Lagrangian fibration on the O'Grady examples, see section 4.5.

We conclude that every known irreducible holomorphic symplectic manifold can be deformed to one which admits a Lagrangian fibration. This is a well known result for K3 surfaces.

2.2. Isotropic line bundles

Lagrangian fibrations and isotropic line bundles on an irreducible holomorphic symplectic manifold X are closely related as for K3 surfaces. One expects many similarities in higher dimensions.

Definition 2.2.1 Let X be an irreducible holomorphic symplectic manifold and L a holomorphic line bundle on X.

- (i) L is called *isotropic*, if the first Chern class $c_1(L)$ is isotropic in the lattice $(H^2(X,\mathbb{Z}), q_X)$ i.e. its Beauville–Bogomolov square $q_X(L,L) = (L,L) = 0$ vanishes.

(*ii*) L is called *nef* (or *numerically effective*), if $c_1(L)$ is contained in the closure $\overline{\mathcal{K}}_X$ of the Kähler cone $\mathcal{K}_X$ in $H^{1,1}(X,\mathbb{R})$. Accordingly, $\overline{\mathcal{K}}_X$ is also called *nef cone* and elements in it are called *nef classes*.

Remark 2.2.2 The classical definition of nefness is the following, see [**Laz04**, 1.4.A]: A line bundle L on a complete variety X is called nef, if

$$C \cdot L \;=\; \int_C c_1(L) \;>\; 0$$

for every irreducible curve $C \subset X$. For a non–algebraic complex space X the definition can be meaningless since X might not contain any curves, for instance certain non–algebraic complex tori.

On a compact hermitian manifold (X, g) there is a metric characterization of nefness: a line bundle L on X is called nef if for every $\epsilon > 0$ there is a hermitian metric h such that the curvature R_h of the associated Chern connection ∇ of h satisfies $R_h \geq -\epsilon g$, cf. [**DPS94**].

However, for an irreducible holomorphic symplectic manifold X there is a description of the nef cone $\overline{\mathcal{K}}_X$ by D. Huybrechts [**Huy03**, Prop. 3.2]:

$$\overline{\mathcal{K}}_X \;=\; \left\{\alpha \in \overline{\mathcal{C}}_X \mid \int_C \alpha \geq 0 \text{ for every rational curve } C \subset X\right\} \tag{2.2.3}$$

and by S. Boucksom [**Bou01**] the actual Kähler cone can be obtained as

$$\mathcal{K}_X \;=\; \left\{\alpha \in \mathcal{C}_X \mid \int_C \alpha > 0 \text{ for every rational curve } C \subset X\right\} \tag{2.2.4}$$

where the rational curves $C \subset X$ might be singular (different as for K3 surfaces where one only has to check the condition with smooth rational curves, cf. [**Huy03**, Cor. 3.4]). In this sense, Definition 2.2.1 always makes sense and coincides with the general definition in the projective case.

The following Lemma should be well known, however we recall it for the reader's convenience.

Lemma 2.2.5 *Let $f : X \to B$ be a Lagrangian fibration and let $L := f^\star A$ be the pullback of a line bundle A on B.*

(*i*) *L is isotropic with respect to the Beauville–Bogomolov quadratic form.*

(*ii*) *If A admits nontrivial sections then L is nef.*

Proof: (*i*) By Fujiki's relation, cf. [**GHJ03**, Prop. 3.9] or Theorem 1.2.2

$$c \cdot q_X(L)^n \;=\; \int_X c_1(L)^{2n} \;=\; \int_X f^\star(c_1(A))^{2n} \;=\; \int_B c_1(A)^{2n} \;=\; 0$$

since $f : X \to B$ is holomorphic (hence orientable) and $c_1(A)^{2n} = 0$. As $c \neq 0$ we have $q_X(L) = 0$.

(*ii*) The pullback L is an effective divisor class, hence $c_1(L)$ belongs to the boundary of the positive cone $\mathcal{C}_X$. By Theorem 2.1.3 we know that $\rho(B) = 1$, therefore A is ample (hence nef, cf. [**Laz04**, 1.4.1]), since A admits a nontrivial section. Therefore, if $C \subset X$ is a curve then $L \cdot C = \deg(f^\star A|_C) \geq$

0, in particular for any rational curve C. By equation (2.2.3) (cf. [**Huy03**, Prop. 3.2]) this implies that $c_1(L)$ is contained in the closure of the Kähler cone i.e. it is nef. $\square$

Recall the following classical result on the existence of fibrations on K3 surfaces.

Theorem 2.2.6 *Let S be a K3 surface.*

(*i*) [**Huy15**, Ch. 2., Prop 3.10] *Let L be a nontrivial isotropic and nef line bundle on S, then there exists an elliptic fibration $f : S \to \mathbb{P}^1$ such that $L = f^\star \mathcal{O}_{\mathbb{P}^1}(1)$.*

(*ii*) [**Huy15**, Ch. 8, Rem. 2.13] *The K3 surface S is elliptic if and only if there is a nontrivial isotropic line bundle on S.*

If L is a line bundle on X we denote by $\varphi_L : X \to |L|$ the induced map by the linear system $|L|$.

Definition 2.2.7 Let X be an irreducible holomorphic symplectic manifold.

(*i*) A *rational Lagrangian fibration* is a dominant meromorphic map $f : X \dashrightarrow B$ such that there exists a bimeromorphic map $\phi : X \dashrightarrow X'$ such that $f \circ \phi^{-1} : X' \to B$ is a Lagrangian fibration.

(*ii*) A line bundle L on X *induces a (rational) Lagrangian fibration* if $\varphi_L : X \to |L|$ defines a (rational) Lagrangian fibration i.e. $\varphi_L : X \to \operatorname{im}(\varphi_L) \subset |L|$ is a (rational) Lagrangian fibration. In particular $2 \dim \operatorname{im}(f) = \dim X$.

(*iii*) The *birational Kähler cone* $\mathcal{BK}_X$ is the union

$$\bigcup_{\phi : X \dashrightarrow X'} \phi^\star \mathcal{K}_{X'}$$

over all bimeromorphic maps $\phi : X \dashrightarrow X'$ where X' is another irreducible holomorphic symplectic manifold.

Remark 2.2.8 Note that in the literature there are slightly different notions of saying L *induces a (rational) Lagrangian fibration*:

- In [**Mar14**, Def. 1.2] it means that $h^0(X, L) = n+1$ where $\dim X = 2n$, $|L|$ is base point free and $\varphi_L : X \to |L|$ is a Lagrangian fibration, in particular a surjective map.
- In [**Mat13**, Def. 1.2] it means as in our Definition 2.2.7 that $\varphi_L : X \to |L|$ is not necessarily surjective, but is a Lagrangian fibration $X \to \operatorname{im}(\varphi_L)$ on its image.

One would like to have a similar result as Theorem 2.2.6 in higher dimensions. One expects the following.

Conjecture 2.2.9 (Huybrechts [**GHJ03**, 21.4], Sawon [**Saw03**, 4.1]) *Let X be an irreducible holomorphic symplectic manifold and L a nontrivial isotropic line bundle on X.*

(i) *If* $c_1(L) \in \mathcal{BK}_X$*, then* L *induces a rational Lagrangian fibration.*

(ii) *If* L *is nef, then* L *induces a Lagrangian fibration.*

Note that the first statement implies the second statement, see Lemma 2.2.14.

Let X be an irreducible holomorphic symplectic manifold and L a line bundle on X. Denote by $\mathcal{L}$ the universal line bundle of the Kuranishi family $\mathfrak{X}_L \to \mathrm{Def}(X, L)$ of the pair (X, L) and D a representative of $\mathrm{Def}(X, L)$. Note that $L = \mathcal{L}_o$ for the reference point o. D. Matsushita [**Mat13**] considers the following subsets of D,

$$\mathrm{D}_{\mathrm{mov}} = \{t \in D \mid c_1(\mathcal{L}_t) \in \mathcal{BK}_X\} \ , \tag{2.2.10}$$

$$\mathrm{D}_{\mathrm{rat}} = \{t \in D \mid \mathcal{L}_t \text{ induces a rational Lagrangian fibration}\} \ , \tag{2.2.11}$$

$$\mathrm{D}_{\mathrm{reg}} = \{t \in D \mid \mathcal{L}_t \text{ induces a Lagrangian fibration}\} \tag{2.2.12}$$

and proves the following.

Theorem 2.2.13 (Matsushita, [**Mat13**, Thm. 1.2]) *Keep the notation of above and assume* L *to be isotropic. Then* $\mathrm{D}_{\mathrm{rat}} = \emptyset$ *or* $\mathrm{D}_{\mathrm{rat}} = \mathrm{D}_{\mathrm{mov}}$*. If the latter is the case, then* $\mathrm{D}_{\mathrm{reg}}$ *is a dense open subset of* D *and* $D \setminus \mathrm{D}_{\mathrm{reg}}$ *is contained in an union of countably many hypersurfaces of* D*.*

As a corollary one gets in combination with with E. Markman's result [**Mar14**, Thm. 1.3, Rmk. 1.8] and K. Yoshioka's result [**Yos12**, Appendix] an answer to Conjecture 2.2.9 for the $\mathrm{K3}^{[n]}$ and generalized Kummer case.

Lemma 2.2.14 [**Mat13**, Claim 3.2] *Keep the notation of above. If the reference point* o *belongs to the closure of* $\mathrm{D}_{\mathrm{reg}}$ *and* L *is nef, then* L *induces a Lagrangian fibration. In particular, if* L *is nef and induces a rational Lagrangian fibration, then it induces a Lagrangian fibration.*

Proof: The first statement is precisely [**Mat13**, Claim 3.2]. We know that L induces a rational Lagrangian fibration i.e. the reference point o is in $\mathrm{D}_{\mathrm{rat}} \subset D$, so $\mathrm{D}_{\mathrm{rat}}$ is nonempty and by Theorem 2.2.13 ([**Mat13**, Thm. 1.2]) $\mathrm{D}_{\mathrm{rat}} = \mathrm{D}_{\mathrm{mov}}$ and $\overline{\mathrm{D}}_{\mathrm{rat}} = \overline{\mathrm{D}}_{\mathrm{reg}} = D$, hence we can apply the first statement. □

Theorem 2.2.15 [**Mat13**, Cor. 1.1] *Let* X *be an irreducible holomorphic symplectic manifold of* $\mathrm{K3}^{[n]}$*–type* [**Mar14**, Thm. 1.3, Rmk. 1.8] *or of generalized Kummer type* [**Yos12**, Appendix]*, let* L *be a nontrivial isotropic line bundle on* X *and assume that* $c_1(L)$ *belongs to the birational Kähler cone* $\mathcal{BK}_X$*.*

(i) *Then* L *induces a rational Lagrangian fibration.*

(ii) *If* L *is addtionally nef, then* L *induces a Lagrangian fibration.*

Proof: The first rational part is the precise statement of [**Mat13**, Cor. 1.1] and the second statement is Lemma 2.2.14. □

Corollary 2.2.16 *Let $f : X \to \mathbb{P}^n$ be a Lagrangian fibration of $K3^{[n]}$–type or of generalized Kummer type. Then $L \coloneqq f^\star \mathcal{O}_{\mathbb{P}^n}(1)$ is primitive.*

Proof: If $L = f^\star \mathcal{O}_{\mathbb{P}^n}(1)$ is not primitive, then write $L = kL'$ with $k > 1$ and L' primitive. The line bundle L' is isotropic and nef since L is isotropic and nef by Lemma 2.2.5. By Theorem 2.2.15 above, the induced map $\varphi_{|L'|}\colon X \to |L'| = \mathbb{P}^n$ by $|L'|$ is a Lagrangian fibration. Clearly we have $L' = \varphi_{|L'|}^\star \mathcal{O}_{\mathbb{P}^n}(1)$, hence $L = \varphi_{|L'|}^\star \mathcal{O}_{\mathbb{P}^n}(k)$. Since $\varphi_{|L'|}$ is surjective, we get

$$n+1 \;=\; h^0(X,L) \;\geq\; h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) \;=\; \binom{n+k}{n} \;>\; n+1\,,$$

a contradiction. $\square$

Remark 2.2.17 We expect $L \coloneqq f^\star A$ to be primitive as long A is primitive for every Lagrangian fibration $f : X \to B$. Note that we have $\rho(B) = 1$ by Theorem 2.1.3. The author does not know a proof for the general case, but the question seems related to Conjecture 2.2.9. If the Lagrangian fibration $f : X \to B$ admits a section $s : B \to X$ then $s^\star L = (f \circ s)^\star A = A$, hence L must be primitive when A is primitive.

A further consequence of Matsuhita's result is the following.

Lemma 2.2.18 [**Mat13**] *Let $f : X \to \mathbb{P}^n$ be a Lagrangian fibration and let $L \coloneqq f^\star A$ be the pullback of an ample line bundle A on $\mathbb{P}^n$. If $\mathcal{L}$ is the universal line bundle of the Kuranishi family $\mathfrak{X}_L \to \mathrm{Def}(X,L)$ and D a representative of $\mathrm{Def}(X,L)$, then the nef locus*

$$\mathrm{D}_{\mathrm{nef}} \;=\; \{t \in D \mid \mathcal{L}_t \text{ is nef}\}$$

and the Lagrangian locus $\mathrm{D}_{\mathrm{reg}}$, see (2.2.12), coincide and are open and dense in D.

Proof: By Theorem 2.2.13 (cf. [**Mat13**, Thm. 1.2]) the locus $\mathrm{D}_{\mathrm{reg}}$ is open and dense in D. By Lemma 2.2.5 (ii) we have $\mathrm{D}_{\mathrm{reg}} \subset \mathrm{D}_{\mathrm{nef}}$. Since $\mathrm{D}_{\mathrm{rat}}$ is nonempty by assumption we have $\overline{\mathrm{D}}_{\mathrm{mov}} = \overline{\mathrm{D}}_{\mathrm{rat}} = \overline{\mathrm{D}}_{\mathrm{reg}} = D$ by Theorem 2.2.13. Therefore $t \in \mathrm{D}_{\mathrm{nef}}$ implies $t \in \overline{\mathrm{D}}_{\mathrm{reg}}$ and we can apply Lemma 2.2.14 to see $t \in \mathrm{D}_{\mathrm{reg}}$. $\square$

2.3. Deformations

We give a precise notion of a *family of Lagrangian fibrations* and relate it to the notion of *deformations of pairs*, see also Proposition 2.4.8. The notion of the latter can be found in [**Mar13**, 5.2].

Definition 2.3.1 (*i*) A *family of Lagrangian fibrations* over a connected complex space S with finitely many irreducible components is an S–morphism

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "\phi"] \arrow[dr] & & P \arrow[dl] \\ & S & \end{tikzcd}$$

where $\mathcal{X} \to S$ is a family of irreducible holomorphic symplectic manifolds and $P \to S$ is a family of projective varieties such that for every $s \in S$ the restriction $\phi|_{\mathcal{X}_s} : \mathcal{X}_s \to P_s$ to the irreducible homorphic symplectic manifold $\mathcal{X}_s$ is a Lagrangian fibration.

(ii) Two Lagrangian fibrations f_1 and f_2 are *deformation equivalent* if there is a family of Lagrangian fibrations ϕ over a connected complex space S containing f_1 and f_2 i.e. there are points $t_i \in S$ such that $\phi_{t_i} = f_i$, $i = 1, 2$.

The above definition seems to be natural. If $f : X \to \mathbb{P}^n$ is a Lagrangian fibration, then the pullback $L := f^\star \mathcal{O}_{\mathbb{P}^n}(1)$ defines a pair (X, L). Therefore it makes sense to consider deformation classes of such pairs and it turns out that the deformation class of the Lagrangian fibration $f : X \to \mathbb{P}^n$ is encoded in the pair (X, L), cf. Proposition 2.4.8.

Proposition 2.3.2 *Let $f_i : X_i \to \mathbb{P}^n$, $i = 1, 2$, denote two Lagrangian fibrations and set $L_i := f_i^\star \mathcal{O}_{\mathbb{P}^n}(1)$. If the Lagrangian fibrations f_i are deformation equivalent in sense of* Definition 2.3.1*, then the pairs (X_i, L_i) are deformation equivalent.*

Proof: Consider a family of Lagrangrian fibrations $\phi : \mathcal{X} \to P$ over a complex space S with points t_i such that $\phi_{t_i} = f_i$ where we can assume that P is a projective bundle as the f_i are fibered over $\mathbb{P}^n$. Let $\pi : \mathcal{X} \to S$ denote the family of irreducible holomorphic symplectic manifolds belonging to the family ϕ. Let $\mathcal{L} := \phi^\star \mathcal{O}_P(1)$ then $e_t := c_1(\mathcal{L}|_{\mathcal{X}_t})$ is clearly of Hodge type $(1,1)$ everywhere and defines a section of $R^2\pi_\star\mathbb{Z}$ such that $e_{t_i} = L_i$ hence the pairs (X_i, L_i) are deformation equivalent. $\square$

2.4. Moduli of Lagrangian fibrations

This section has the purpose to explain what we mean by the *moduli space of Lagrangian fibrations*. For the K3$^{[n]}$–type we describe connected components of it. Many of the constructions and explanations can be found in [**Mar11**], [**Mar14**] and [**Mat13**].

D. Matsushita [**Mat09**] constructed a *local moduli space* for arbitrary Lagrangian fibrations.

Let $\pi : \mathfrak{X} \to \mathrm{Def}(X)$ denote the Kuranishi family of an irreducible holomorphic symplectic manifold $X = \pi^{-1}(0)$. For a line bundle L on X let $\pi : \mathfrak{X}_L \to \mathrm{Def}(X, L)$ denote the universal family of the pair (X, L). Further denote by $\mathcal{L}$ the universal line bundle on $\mathfrak{X}_L$ and set $\mathcal{L}_t := \mathcal{L}|_{\mathfrak{X}_{L,t}}$.

Theorem 2.4.1 [**Mat09**, Cor. 1.3] *Let $f : X \to B$ be a Lagrangian fibration and L be the pullback of a very ample line bundle on B. Then $\mathcal{L}$ is a π–relatively base point free line bundle i.e. after shrinking the representative* $\mathrm{Def}(X, L)$ *there exists a family* *of Lagrangian fibrations*

$$\begin{tikzcd}
\mathfrak{X}_L \arrow[rr, "\zeta"] \arrow[dr, "\pi"'] & & \mathbb{P}(\pi_\star \mathcal{L}) \arrow[dl] \\
& \mathrm{Def}(X,L) &
\end{tikzcd}$$

over $\mathrm{Def}(X,L)$ *such that* $\zeta_0 = f$.

2.4.2. The general moduli space of Lagrangian fibrations. Matsushita's result allows to construct a *global moduli space*. Let Λ be an abstract lattice which is isometric to the second cohomology of an irreducible holomorphic symplectic manifold and denote by $\mathfrak{M}_\Lambda$ the associated moduli of marked pairs.

We can glue all total spaces $\mathrm{Def}(X,L)$ of such families i.e. $\mathfrak{X}_L \to \mathrm{Def}(X,L)$ for $f : X \to \mathbb{P}^n$ a Lagrangian fibration and L a line bundle on X as in Theorem 2.4.1 with $H^2(X,\mathbb{Z})$ isometric to Λ, similar to the construction of the moduli of marked pairs, cf. 1.3.5. As formula:

$$\mathfrak{h}_\Lambda \;:=\; \bigsqcup_{\substack{(X,\eta)\in\mathfrak{M}_\Lambda \\ \mathfrak{X}_L \to \mathrm{Def}(X,L)}} \mathrm{Def}(X,L)/\sim$$

with $x_1 \sim x_2$ for $x_i \in \mathrm{Def}(X_i, L_i)$ if there is a biholomorphism between neighborhoods of the x_i respecting the extended markings. As sets we have

$$\mathfrak{h}_\Lambda \;=\; \{(X,\eta) \in \mathfrak{M}_\Lambda \mid \text{there exists a Lagrangian fibration on } X\}$$

The result is a possibly non–Hausdorff moduli space $\mathfrak{h}_\Lambda$ of Lagrangian fibrations of deformation type X' with $\dim \mathfrak{h}_\Lambda = b_2(X) - 3$. It is not clear, that $\mathfrak{h}_\Lambda$ embeds globally into the moduli space of marked irreducible holomorphic symplectic manifolds $\mathfrak{M}_\Lambda$, but clearly it is locally a smooth submanifold of codimension one of $\mathfrak{M}_\Lambda$.

2.4.3. A connected component. In the $\mathrm{K3}^{[n]}$ and generalized Kummer case, results of E. Markman [**Mar13**], [**Mar14**] and D. Matsushita [**Mat13**], [**Mat09**] provide methods to describe a connected component of the moduli space of Lagrangian fibrations.

Let Λ denote a lattice of signature $(3, b_2 - 3)$ which is isometric to the second integral cohomology of an irreducible holomorphic symplectic manifold.

Let $\mathfrak{M}_\Lambda$ denote the corresponding moduli space of isomorphism classes of marked pairs (X,η) i.e. X is an irreducible holomorphic symplectic manifold of the fixed deformation type and $\eta : H^2(X,\mathbb{Z}) \to \Lambda$ is a marking. Choose a connected component $\mathfrak{M}_\Lambda^\circ$ of $\mathfrak{M}_\Lambda$ and consider the period map

$$\mathcal{P} \;:\; \mathfrak{M}_\Lambda^\circ \longrightarrow \Omega_\Lambda\,, \quad (X,\eta) \longmapsto [\eta(H^{2,0}(X))]\,.$$

Choose the orientation of $\tilde{\mathcal{C}}_\Lambda$ compatible to $\mathfrak{M}_\Lambda^\circ$ in sense of Definition 1.4.16.

Let $\lambda \in \Lambda$ be a nontrivial isotropic class. After a possible change of the sign of λ (cf. 1.4.19), we have a distinguished and compatible connected component

$$\Omega^{+}_{\lambda^\perp} \ := \ \{p \in \Omega_{\lambda^\perp} \mid \lambda \in \partial \mathcal{C}_p\}$$

of the hyperplane section $\Omega_{\lambda^\perp} = \Omega_\Lambda \cap \lambda^\perp$, see 1.4.19. Then define

$$\mathfrak{M}^\circ_{\lambda^\perp} \ := \ \mathcal{P}^{-1}\left(\Omega^{+}_{\lambda^\perp}\right) \ = \ \left\{(X,\eta) \in \mathfrak{M}^\circ_\Lambda \mid \eta^{-1}(\lambda) \text{ is of type } (1,1) \text{ and in } \partial\mathcal{C}_X \right\}.$$

Lemma 2.4.4 ([**Mar14**, Lem. 4.4], [**Mar13**, Cor. 5.11]) *The space $\mathfrak{M}^\circ_{\lambda^\perp}$ is a connected hypersurface of $\mathfrak{M}^\circ_\Lambda$.*

Consider the nef subspace

$$\mathfrak{U}^\circ_{\lambda^\perp} \ := \ \left\{(X,\eta) \in \mathfrak{M}^\circ_{\lambda^\perp} \mid \eta^{-1}(\lambda) \text{ is nef}\right\}.$$

We claim that this space $\mathfrak{U}^\circ_{\lambda^\perp}$ is a connected component of the moduli space of Lagrangian fibrations of the fixed deformation type, see Theorem 2.4.7 below. Furthermore it is connected and open in $\mathfrak{M}^\circ_{\lambda^\perp}$.

As a consequence of Lemma 2.2.18 the space $\mathfrak{U}^\circ_{\lambda^\perp}$ is locally isomorphic to $\mathrm{Def}(X,L)$.

Proposition 2.4.5 *Let (X,η) be a marked irreducible holomorphic symplectic manifold, $f : X \to \mathbb{P}^n$ be a Lagrangian fibration on X and $L = f^\star\mathcal{O}_{\mathbb{P}^n}(1)$. Set $\lambda := \eta(c_1(L))$. Then $\mathfrak{U}^\circ_{\lambda^\perp}$ and $\mathrm{Def}(X,L)$ are locally isomorphic around (X,η) and 0 respectively.*

Proof: Note that λ is isotropic by Lemma 2.2.5. Under the assumption that $\mathrm{Def}(X)$ is chosen sufficiently small there exists an unique extension $\Sigma : R^2\pi_\star\mathbb{Z} \to \Lambda_{\mathrm{Def}(X)}$ of η i.e. $\Sigma_0 = \eta$ and we have a local isomorphism $F : \mathrm{Def}(X) \to \mathfrak{M}_\Lambda$ by the construction of the moduli of marked pairs. More precisely it is given by

$$F : \mathrm{Def}(X) \longrightarrow \mathfrak{M}_\Lambda\,, \quad t \longmapsto (\mathfrak{X}_t, \Sigma_t)$$

and we have the following diagram

$$\begin{tikzcd} \mathrm{Def}(X) \arrow[d, "F"'] & \mathrm{Def}(X,L) \arrow[l, hook'] \arrow[d, "F"] \\ \mathfrak{M}_\Lambda & \mathfrak{M}^\circ_{\lambda^\perp} \arrow[l, hook'] \end{tikzcd}$$

By Lemma 2.2.18 we can choose $\mathrm{Def}(X,L)$ small such that $\mathcal{L}_t$ is nef for every $t \in \mathrm{Def}(X,L)$. If we restrict F to $\mathrm{Def}(X,L)$ it takes values in $\mathfrak{U}^\circ_{\lambda^\perp}$: For $t \in \mathrm{Def}(X,L)$ the mapping $t \mapsto c_1(\mathcal{L}_t)$ is a section of $R^2\pi_\star\mathbb{Z}|_{\mathrm{Def}(X,L)}$ so in particular $t \mapsto \Sigma_t(c_1(\mathcal{L}_t)) \in \Lambda$ is continuous. Hence it is constant as Λ is discrete and $\mathrm{Def}(X,L)$ sufficiently small. This implies that $\Sigma_t(c_1(\mathcal{L}_t)) = \Sigma_t(L) = \lambda$ so $\Sigma_t^{-1}(\lambda) = c_1(\mathcal{L}_t)$ is nef i.e. $F(t) \in \mathfrak{U}^\circ_{\lambda^\perp}$. Since $\mathrm{Def}(X,L)$ and $\mathfrak{M}^\circ_{\lambda^\perp}$ are hypersurfaces in $\mathrm{Def}(X)$ and $\mathfrak{M}_\Lambda$ respectively $F(\mathrm{Def}(X,L))$ is an open set in $\mathfrak{M}^\circ_{\lambda^\perp}$ and contained in $\mathfrak{U}^\circ_{\lambda^\perp}$. Hence $\mathrm{Def}(X,L)$ and $\mathfrak{U}^\circ_{\lambda^\perp}$ are locally isomorphic. □

Corollary 2.4.6 *The space $\mathfrak{U}^\circ_{\lambda^\perp}$ is smooth of dimension $\mathrm{rk}\,\Lambda - 3$ and open in $\mathfrak{M}^\circ_{\lambda^\perp}$.*

Proof: By Proposition 2.2.18 and Proposition 2.4.5 it is open in $\mathfrak{M}^\circ_{\lambda^\perp}$. Smoothness was also shown in Proposition 2.4.5. $\square$

We now restrict to the $\mathrm{K3}^{[n]}$ or generalized Kummer deformation type. Then we can summarize the discussion as the following.

Theorem 2.4.7 *Let λ be a primitive and isotropic element in the $K3^{[n]}$ or generalized Kummer lattice. The space $\mathfrak{U}^\circ_{\lambda^\perp}$ in the corresponding connected component $\mathfrak{M}^\circ_\Lambda$ of the moduli of marked pairs has the following properties.*

(i) *It parametrizes isomorphism classes of marked pairs (X,η) of $\mathfrak{M}^\circ_\Lambda$ with X of $K3^{[n]}$ or generalized Kummer type, respectively, admitting a Lagrangian fibration $f : X \to \mathbb{P}^n$ such that*

$$\eta\left(c_1\left(f^\star\mathcal{O}_{\mathbb{P}^n}(1)\right)\right) \;=\; \lambda\,.$$

(ii) *It is smooth of dimension 20 for the $K3^{[n]}$ and of dimension 4 for the generalized Kummer case. Further it is open in $\mathfrak{M}^\circ_{\lambda^\perp}$.*

(iii) *It is connected.*

Proof: (i) Let $(X,\eta) \in \mathfrak{U}^\circ_{\lambda^\perp}$. As $H^1(X,\mathcal{O}_X) = 0$ the exponential sequence on X is

$$\cdots 0 \longrightarrow \mathrm{Pic}(X) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathcal{O}_X) \longrightarrow \cdots .$$

Since $\eta^{-1}(\lambda)$ is of type $(1,1)$ it is in the kernel of $H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X)$ so we can find an unique line bundle L on X such that $c_1(L) = \eta^{-1}(\lambda)$. Then by Theorem 2.2.15 L induces a Lagrangian fibration $f : X \to |L^\star| = \mathbb{P}^n$ since $\eta^{-1}(\lambda)$ is nef by assumption.

Conversely let (X,η) be a marked pair with X of $\mathrm{K3}^{[n]}$ or generalized Kummer type admitting a Lagrangian fibration $f : X \to \mathbb{P}^n$. Then $\lambda := \eta(c_1(f^\star\mathcal{O}_{\mathbb{P}^n}(1)))$ is primitive by Corollary 2.2.16 and isotropic by Lemma 2.2.5 (i). We then have $\mathcal{P}(X,\eta) \in \Omega^+_{\lambda^\perp}$ i.e. (X,η) is in $\mathfrak{M}^\circ_{\lambda^\perp}$. In particular $\eta^{-1}(\lambda) = c_1(f^\star\mathcal{O}_{\mathbb{P}^n}(1))$ is nef by Lemma 2.2.5 (ii). This implies that (X,η) is in $\mathfrak{U}^\circ_{\lambda^\perp}$.

(ii) This was content of Corollary 2.4.6, as $\mathrm{rk}\,\Lambda = 23$ or $\mathrm{rk}\,\Lambda = 7$ in the $\mathrm{K3}^{[n]}$ or generalized Kummer case, respectively.

(iii) Let (X,η) be in $\mathfrak{M}^\circ_{\lambda^\perp} \setminus \mathfrak{U}^\circ_{\lambda^\perp}$, then by definition $\eta^{-1}(\lambda)$ is not nef. By [**Rie14**, Prop. 3.14.] there exists a monodromy operator $g \in \mathrm{Mon}^2(X)$ such that g preserves the Hodge structure of $H^2(X,\mathbb{Z})$ and $g(\eta^{-1}(\lambda)) \in \overline{\mathcal{BK}}_X$ where the latter denotes the closure of the birational Kähler cone $\mathcal{BK}_X$. By [**MY15**, Cor. 1.5] there exists a bimeromorphic map $\phi : X \to X'$ where X' is irreducible holomorphic symplectic such that $g(\eta^{-1}(\lambda)) = \phi^\star\alpha$ where $\alpha \in \overline{\mathcal{K}}_X$ is nef on X'. Then $\eta' := \eta \circ g^{-1} \circ \phi^\star$ is a marking on X' and $\eta'^{-1}(\lambda) = \alpha$ is nef, hence the pair (X',η') is contained in $\mathfrak{U}^\circ_{\lambda^\perp}$. Since g preserves the Hodge structure we have in particular $\mathcal{P}(X,\eta) = \mathcal{P}(X',\eta')$

for the periods. By M. Verbitsky's Global Torelli Theorem 1.3.8 ([**Ver13**, Thm. 4.24]) the pairs (X,η) and (X',η') are inseparable points of $\mathfrak{M}^0_\Lambda$. This shows that the Hausdorffization of $\mathfrak{M}^0_{\lambda^\perp}$ coincide with the Hausdorffization of $\mathfrak{U}^0_{\lambda^\perp}$ which therefore must be connected since $\mathfrak{M}^\circ_{\lambda^\perp}$ is connected. We conclude that $\mathfrak{U}^0_{\lambda^\perp}$ is connected as its Hausdorffization is. □

Note that in the proof of (i) and (iii) we have used statements that are up to now only known for the K3$^{[n]}$ or generalized Kummer deformation type.

We can now extend Proposition 2.3.2 for the K3$^{[n]}$ or generalized Kummer case. We deal with the question: when do two Lagrangian fibrations of K3$^{[n]}$ or generalized Kummer type lie in the same connected component $\mathfrak{U}^\circ_{\lambda^\perp}$?

Proposition 2.4.8 *Let $f_i : X_i \to \mathbb{P}^n$, $i = 1, 2$, denote two Lagrangian fibrations with both of K3$^{[n]}$–type or both of generalized Kummer type. Accordingly let Λ denote the K3$^{[n]}$–lattice or the generalized Kummer lattice respectively and set $L_i := f_i^\star \mathcal{O}_{\mathbb{P}^n}(1)$. Then the following statements are equivalent.*

(*i*) *The Lagrangian fibrations f_i are deformation equivalent in sense of* Definition 2.3.1.

(*ii*) *The pairs (X_i, L_i) are deformation equivalent.*

(*iii*) *There exist markings $\eta_i : H^2(X_i,\mathbb{Z}) \to \Lambda$ such that*

$$\eta_1(c_1(L_1)) \;=\; \eta_2(c_1(L_2))$$

and $\eta_2^{-1} \circ \eta_1$ is a parallel transport operator.

(*iv*) *There exist markings $\eta_i : H^2(X_i,\mathbb{Z}) \to \Lambda$ such that the marked pairs (X_i,η_i) are contained in the same connected component $\mathfrak{U}^\circ_{\lambda^\perp}$ for a primitive isotropic class λ in the K3$^{[n]}$ or generalized Kummer lattice.*

Proof: By Proposition 2.3.2 we only need to show (ii) ⇒ (iii), (iii) ⇒ (iv) and (iv) ⇒ (i).

(ii) ⇒ (iii) Let $\pi : \mathcal{X} \to S$ be a family of irreducible holomorphic symplectic manifolds with S connected, t_i points such that $\mathcal{X}_{t_i} = X_i$ and e a section of $R^2\pi_\star\mathbb{Z}$ with $e_{t_i} = c_1(L_i)$. As $R^2\pi_\star\mathbb{Z}$ is a local system we can find a neighbourhood U of t_2 and a marking $\Sigma : R^2\pi_\star\mathbb{Z}|_U \to \Lambda_U$. As S is connected we can choose a path γ connecting t_1 with t_2. Then γ is parallel along e i.e. $\gamma^\star e$ is a flat section of $\gamma^\star R^2\pi_\star\mathbb{Z}$. Consider the parallel transport $P_\gamma : H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$ along γ. Note that $P_\gamma(e_{t_1}) = e_{t_2}$. Define $\eta_2 := \Sigma_{t_2}$ and $\eta_1 := \eta_2 \circ P_\gamma$. Hence we have $\eta_2^{-1} \circ \eta_1 = P_\gamma$ and

$$\eta_1(c_1(L_1)) \;=\; \eta_1(e_{t_1}) \;=\; \eta_2(P_\gamma(e_{t_1})) \;=\; \eta_2(e_{t_2}) \;=\; \eta_2(c_1(L_2))\,.$$

(iii) ⇒ (iv) As $\eta_2^{-1} \circ \eta_1$ is a parallel transport operator the manifolds X_i belong to a family of irreducible holomorphic symplectic manifolds. Hence the marked pairs (X_i,η_i) belong to a connected component $\mathfrak{M}^\circ_\Lambda$ of the moduli of marked pairs. The condition that $\lambda := \eta_1(c_1(L_1)) = \eta_2(c_1(L_2))$ then implies that the pairs (X_i,η_i) are contained in $\mathfrak{M}^{\circ}_{\lambda^{\perp}}$, but also in $\mathfrak{U}^{\circ}_{\lambda^{\perp}}$ since the L_i are nef and $\mathfrak{U}^{\circ}_{\lambda^{\perp}}$ is connected by Theorem 2.4.7 (iii).

(iv) $\Rightarrow$ (i) Choose a path γ in $\mathfrak{U}^{\circ}_{\lambda^{\perp}}$ connecting the marked pairs (X_i, η_i). We can choose finitely many points $x_1, \ldots, x_N$ which lie on γ with the following properties

- $(X_1, \eta_1) = x_1$ and $(X_2, \eta_2) = x_N$
- By Theorem 2.4.7 each x_k corresponds to a Lagrangian fibration $f_k : X_k \to \mathbb{P}^n$ which belongs by Theorem 2.4.1 to a family of Lagrangian fibrations $\zeta_k : \mathfrak{X}_k \to P_k$ parametrised by $\mathrm{Def}_k := \mathrm{Def}(X_k, f_k^{\star}\mathcal{O}_{\mathbb{P}^n}(1))$. Therefore each x_k admits the neighbourhood Def_k with $\mathrm{Def}_k \cap \mathrm{Def}_{k+1}$ nonempty for $k = 1, \ldots, N-1$.
- Note that γ is covered by the Def_k, $k = 1, \ldots, N$.

Choose points in z_k in $\mathrm{Def}_k \cap \mathrm{Def}_{k+1}$ for $k = 1, \ldots, N-1$.

- Set $S := \coprod_{k=1}^{N} \mathrm{Def}_k / \sim$ where $\sim$ glues Def_k and Def_{k+1} at the point z_k for $k = 1, \ldots, N-1$.
- Further set $\mathcal{X} := \coprod_{k=1}^{N} \mathfrak{X}_k / \sim$ where $\sim$ glues $\mathfrak{X}_k$ and $\mathfrak{X}_{k+1}$ at $(\mathfrak{X}_k)_{z_k}$ and $(\mathfrak{X}_{k+1})_{z_k}$. Note that those fibers are isomorphic.
- Denote by $\pi_k : \mathfrak{X}_k \to \mathrm{Def}_k$ the family of irreducible holomorphic symplectic manifolds belonging to the family ζ_k. Then the map $\pi : \mathcal{X} \to S$ defined by $\pi|_{\mathfrak{X}_k} := \pi_k$ is well defined and is a family of irreducible holomorphic symplectic manifolds.
- Set $P := \coprod_{k=1}^{N} P_k / \sim$ where $\sim$ glues P_k and P_{k+1} at the projective spaces $(P_k)_{z_k}$ and $(P_{k+1})_{z_{k+1}}$ for $k = 1, \ldots, N-1$. We get a morphism $P \to S$ which is induced by the morphisms $P_k \to \mathrm{Def}_k$, $k = 1, \ldots, N$. This map is a family of projective spaces hence a projective bundle.

Putting everything together we can define a map $\phi : \mathcal{X} \to P$ locally given by the $\zeta_k : \mathfrak{X}_k \to P_k$, $k = 1, \ldots, N$. This defines by construction a family of Lagrangian fibrations over S containing $f_1 = \phi_{x_1}$ and $f_2 = \phi_{x_2}$. $\square$

CHAPTER 3

Polarization Types of Lagrangian Fibrations

In this chaper X will always denote an irreducible holomorphic symplectic manifold of dimension $2n$ and $f : X \to B$ a Lagrangian fibration.

For a general point $t \in B$ the associated fiber $F := f^{-1}(t)$ is an abelian variety even when X is not projective. That F is actually projective follows from [**Cam06**, Prop. 2.1], see Proposition 2.1.2. A related statement of the latter is Proposition 3.1.3.

In this chaper we explain how to associate to f a tuple $\underline{\mathrm{d}}(f) \in \mathbb{Z}^n$ of positive integers which is called the *polarization type* of the fibration. This type $\underline{\mathrm{d}}(f)$ is the type of a polarization on F in the classical sense, see Appendix B.1.

3.1. Special Kähler classes

Let $f : X \to B$ a Lagrangian fibration on a irreducible holomorphic symplectic manifolds X of dimension $\dim X = 2n$. A *polarization* on an abelian variety A is by definition the first Chern class $c_1(L) \in H^2(A, \mathbb{Z})$ of an ample line bundle L on A. If X is non–projective, it is not clear how to obtain a polarization of an arbitrary chosen smooth fiber. In the projective case, one can just use an ample line bundle. However, in the general situation *special Kähler classes*[1] will provide us the polarizations.

For this section, fix a smooth fiber F.

Definition 3.1.1 (Special Kähler Class) We say that a Kähler class $\omega \in \mathcal{K}_X \subset H^{1,1}(X, \mathbb{R})$ is *special with respect to F* if the restriction $\omega|_F$ is integral i.e. contained in $H^2(F, \mathbb{Z})$ and primitive i.e. indivisible. We call such an ω just *special* if there is no confusion with the fiber F.

Example 3.1.2 Of course every ample line bundle $L \in \mathrm{Pic}(X)$ defines a Kähler class $c_1(L) \in H^{1,1}(X, \mathbb{Z})$ which is integral on all fibers and for each smooth fiber F we can find a natural number k (which may depend on F) such that $\omega := \frac{1}{k} c_1(\mathcal{L})$ is special with respect to F.

The following Proposition is related to an observation of C. Voisin, see [**Cam06**, Proof of Prop. 2.1][2].

[1] this is not related to special Kähler geometry – the author just has not found a better name.

[2] F. Campana states in an earlier version of his paper, that [**Cam06**, Proof of Prop. 2.1] is due to communication with C. Voisin.

Proposition 3.1.3 *For every smooth fiber F there is a Kähler class ω on X which is special with respect to F.*

Proof: We have a surjective projection $p : H^2(X,\mathbb{R}) \to H^{1,1}(X,\mathbb{R})$ which is induced by the Hodge decomposition. As $H^2(X,\mathbb{Q})$ is dense in $H^2(X,\mathbb{R})$ also $p(H^2(X,\mathbb{Q}))$ is dense in $H^{1,1}(X,\mathbb{R})$. Since the Kähler cone $\mathcal{K}_X$ is open in $H^{1,1}(X,\mathbb{R})$ we have $p(H^2(X,\mathbb{Q})) \cap \mathcal{K}_X \neq \emptyset$ so that we can find a class $\alpha \in H^2(X,\mathbb{Q})$ with $p(\alpha) \in \mathcal{K}_X$. Denote by $r : H^2(X,\mathbb{R}) \to H^2(F,\mathbb{R})$ the restriction. As F is Lagrangian and $H^{2,0}(X)$ is generated by the holomorphic symplectic form the restriction $r_{H^{2,0}} : H^{2,0}(X) \to H^{2,0}(F)$ on holomorphic two–forms is zero, hence the non–$(1,1)$ parts of α are in the kernel of r so we have $r(\alpha) = r(p(\alpha))$. Then take a positive number $m > 0$ such that $mr(\alpha) \in H^2(F,\mathbb{Z})$ is integral and primitive. Consequently $\omega := mp(\alpha)$ is a special Kähler class on X with respect to F since $r(\omega) = mr(p(\alpha)) = mr(\alpha) \in H^2(F,\mathbb{Z})$. $\square$

3.2. The restriction of the Kähler cone is a ray

The restriction $\omega|_F$ of a Kähler class which is special with respect to the smooth fiber F defines a primitive polarization on the abelian variety F. We are therefore interested in the restriction of the Kähler cone to the fiber.

Lemma 3.2.1 *Let $\mathcal{K}_X$ be the Kähler cone of X, F a smooth fiber and $r : H^2(X,\mathbb{C}) \to H^2(F,\mathbb{C})$ the restriction.*

(*i*) *Then* $\operatorname{rk} r = 1$, *and*

(*ii*) $G := r(\mathcal{K}_X) \subset H^{1,1}(F,\mathbb{R})$ *is a ray that contains integral points.*

Proof: (*i*) We consider the Kuranishi family $\pi : \mathfrak{X} \to \mathrm{Def}(X)$. By [**Voi92**, Prop 1.2, Lemma 1.5] the space

$$D_F \;:=\; \{t \in \mathrm{Def}(X) \mid \text{there exists a deformation } \mathcal{F}_t \subset \mathfrak{X}_t \text{ of } F\}\ .$$

is a complex submanifold of $\mathrm{Def}(X)$ and for its codimension in $\mathrm{Def}(X)$ one has $\operatorname{codim} D_F = \operatorname{rk} r$.

Let L be the pullback of a very ample line bundle on B by f. Then $\mathrm{Def}(X,L) \subset D_F$: By Theorem 2.4.1 we have a family $\zeta : \mathfrak{X}_L \to P := \mathbb{P}(\pi_\star L)$ of Lagrangian fibrations over $\mathrm{Def}(X,L)$ where $\pi : \mathfrak{X}_L \to \mathrm{Def}(X,L)$ is the universal family of the pair (X,L) such that $\zeta_0 = f$. Let $F = X_{t_0}$ be the fiber over the point $t_0 \in B = P_0$ of $X \to P_0$. Then we can choose a neighbourhood U of 0 and a local holomorphic section $s : U \to P$ of the $\mathbb{P}^n$–fibration $\mathfrak{X}_L \to P$ such that $s(0) = t_0$. Then the fiber product $\mathcal{F} := U \times_P \mathfrak{X}_L$ gives a deformation $\mathrm{pr}_U : \mathcal{F} \to U$ of $\mathcal{F}_0 = F$ hence $U \subset \mathrm{Def}(X,L)$ i.e. $\mathrm{Def}(X,L) \subset D_F$ as germs. For the codimensions in $\mathrm{Def}(X)$ we have in particular

$$1 = \;\operatorname{codim} \mathrm{Def}(X,L) \;\geq\; \operatorname{codim} D_F \;=\; \operatorname{rk} r \;\geq\; 1$$

as a Kähler class on X restricts to a nontrivial element in $H^2(F,\mathbb{C})$. We conclude that $\operatorname{rk} r = 1$.

(*ii*) By (i) we have also $\mathrm{rk}(r : H^{1,1}(X,\mathbb{R}) \to H^{1,1}(F,\mathbb{R})) = 1$. As $\mathcal{K}_X$ is open in $H^{1,1}(X,\mathbb{R})$ it follows that $\dim G = 1$. Since restrictions of Kähler classes are still Kähler classes, G is a ray. By Proposition 3.1.3 G contains integral points. □

Remark 3.2.2 (*i*) If $\pi : \mathfrak{X} \to \mathrm{Def}(X)$ denotes the Kuranishi family then the local system $R^2\pi_\star\mathbb{C}_{\mathfrak{X}}$ is trivial as we assume $\mathrm{Def}(X)$ to be simply connected. By Ehresmann's theorem we can choose a differentiable trivialization

$$\begin{tikzcd} \mathfrak{X} \arrow[dr] & & X \times \mathrm{Def}(X) \arrow[ll, "\rho"'] \arrow[dl] \\ & \mathrm{Def}(X) & \end{tikzcd}$$

where we denote by $\rho_t := \rho|_X : X \to \mathfrak{X}_t$ the associated fiber diffeomorphism. Further we can choose a relative holomorphic form σ i.e. a section of $\Omega^2_{\mathfrak{X}/\mathrm{Def}(X)}$ such that the restriction $\sigma_t := \sigma|_{\mathfrak{X}_t}$ is a holomorphic symplectic form on $\mathfrak{X}_t$. Then the space D_F in the proof of Lemma 3.2.1 can also be defined as

$$D_F = \left\{ t \in \mathrm{Def}(X) \mid r[\rho_t^\star \sigma_t] = 0 \in H^2(F,\mathbb{C}) \right\},$$

see [**Voi92**, Thm 0.1].

(*ii*) From the proof it also follows that $D_F = \mathrm{Def}(X,L)$ as germs as D_F is irreducible and contained in $\mathrm{Def}(X,L)$, but both have codimension one in $\mathrm{Def}(X)$.

3.3. The associated family of special Kähler classes

Let $\Delta \subset B$ be the *discriminant locus* of the Lagrangian fibration $f : X \to B$, which is by definition the set parametrizing the singular fibers. Note that in general Δ is a reducible hypersurface in B, see [**HO09**, Prop 3.1]. Then $B^\circ := B - \Delta$ is a connected open subset and the restriction $g := f|_{f^{-1}(B^\circ)} : f^{-1}(B^\circ) \to B^\circ$ is a proper holomorphic submersion. Let $\mathcal{C}^\infty_{B^\circ}$ denote the sheaf of smooth real functions on B°. By Ehresmann's theorem $\mathcal{H} := R^2 g_\star\mathbb{R} \otimes \mathcal{C}^\infty_{B^\circ}$ is a differentiable real vector bundle on B° which comes with a canonical flat connection ∇ called the Gauss–Manin connection, see [**Voi02**, 9.2.1].

For each $t \in B^\circ$ consider the restriction $r_t : H^2(X,\mathbb{R}) \to H^2(X_t,\mathbb{R})$ where $X_t := g^{-1}(t)$. Set $\mathcal{G}_t := r_t(\mathcal{K}_X) \cap H^2(X_t,\mathbb{Z})$. By Lemma 3.2.1 $\mathcal{G}_t$ is a non–empty semigroup of rank one. We can define the following map.

Definition 3.3.1 The associated *family of special Kähler classes* is the map

$$\alpha : B^\circ \longrightarrow \mathcal{H} = R^2 g_\star\mathbb{R} \otimes \mathcal{C}^\infty_{B^\circ} \text{ such that} \tag{3.3.2}$$
$$t \longmapsto \alpha(t) \in \mathcal{G}_t \subset H^2(X_t,\mathbb{R})$$

is the unique integral and primitive element in $\mathcal{G}_t$ for all $t \in B^\circ$.

Proposition 3.3.3 *The $\mathcal{G}_t$ form a local system $\mathcal{G}$ of semigroups on B°. The map $\alpha : B^\circ \to \mathcal{H}$ is continuous, and thus can be considered as a section of $\mathcal{G}$.*

Proof: Consider the family of sections

$$\varphi : \mathcal{K}_X \times B^\circ \to \mathcal{H}\,, \quad (\omega, t) \mapsto \omega|_{X_t}\,. \tag{3.3.4}$$

Then the image of φ is the union of rays in each $H^2(X_t, \mathbb{R})$ considered in Lemma 3.2.1 containing integral points. Note that the family φ is differentiable as for each $\omega \in \mathcal{K}_X$ the corresponding section $\varphi(\omega, \cdot)$ is differentiable. More precisely it is flat, i.e. $\nabla\varphi(\omega, \cdot) = 0$ for each $\omega \in \mathcal{K}_X$ which follows from the Cartan–Lie formula, see [**Voi02**, 9.2.2].

Let $\mathcal{H}^\nabla$ be the sheaf of flat sections of $\mathcal{H} = R^2 g_\star \mathbb{R} \otimes \mathcal{C}^\infty_{B^\circ}$. As φ is a flat family the image $\operatorname{im}\varphi$ is a local system of semigroups which is contained in $\mathcal{H}^\nabla$. Then define $\mathcal{G} := \operatorname{im}\varphi \cap R^2 g_\star \mathbb{Z}$ which is in a canonical way a local system whose stalks are given precisely by $\mathcal{G}_t$.

Take an open cover $B^\circ = \cup_i U_i$ such that $\mathcal{G}$ is trivial on each U_i say $\mathcal{G}(U_i) = G$ for all i where $G := \mathcal{G}_t$ for a fixed t. For each i the restriction $\alpha|_{U_i}$ is the unique primitive element in G. They glue to an unique global section of $\mathcal{G}$ which is precisely α. Hence α is continuous as a section of the local system $\mathcal{G}$ and in particular as a map $B^\circ \to \mathcal{H}$. $\square$

3.4. Definition of the polarization type of a Lagrangian fibration

Let α denote the associated family of special Kähler classes to the Lagrangian fibration $f : X \to B$.

Clearly $\alpha(t) \in H^2(X_t, \mathbb{Z})$ defines a polarization on the abelian variety X_t for every $t \in B^\circ$. To any polarization on an abelian variety one can associate a tuple of positive integers which is called the *polarization type*, see Appendix (B.1) and [**BL03**, p. 70].

Following (B.1) we have an identification $H^2(X_t, \mathbb{Z}) = \bigwedge^2 H_1(X_t, \mathbb{Z})^\vee$ and view $\alpha(t) : \Lambda_t \otimes \Lambda_t \to \mathbb{Z}$ as an alternating integral form on the lattice $\Lambda_t := H_1(X_t, \mathbb{Z})$. By the elementary divisor theorem we can find a basis of Λ_t for which $\alpha(t)$ has the form

$$\alpha(t) \;=\; \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$$

where $D = \operatorname{diag}(d_1, \ldots d_n)$ is an integral diagonal matrix with $d_i > 0$ and $d_i | d_{i+1}$. The tuple

$$\underline{\mathrm{d}}(f, t) \;:=\; (d_1, \ldots, d_n)$$

is called the *polarization type* of $\alpha(t)$ and a priori depends on $t \in B^\circ$. Note that we also use the notation $\underline{\mathrm{d}}(L)$ for the type of a polarization L on an abelian variety in the classical sense, cf. (B.1), i.e. $\underline{\mathrm{d}}(f, t) = \underline{\mathrm{d}}(\alpha(t))$.

Proposition 3.4.1 *The polarization type $\underline{\mathrm{d}}(f, \cdot) : B^\circ \to \mathbb{Z}^n$ is constant.*

Proof: By construction for $t \in B^\circ$ the associated tuple $\underline{\mathrm{d}}(f,t)$ is the diagonal of one of the blocks of the representation matrix of $\alpha(t) : \Lambda_t \times \Lambda_t \to \mathbb{Z}$ with respect to a chosen basis $b_1(t), \ldots, b_{2n}(t)$ of the lattice $H_1(X_t, \mathbb{Z})$. This correspondence is continuous and since $\underline{\mathrm{d}}(f,\cdot)$ is integer valued in each component it is locally constant hence constant as B° is connected. $\square$

Definition 3.4.2 (Polarization Type) For each Lagrangian fibration $f : X \to B$ the associated tuple $\underline{\mathrm{d}}(f)$ in $\mathbb{Z}^n$ is called the *polarization type* of f.

Theorem 3.4.3 *The polarization type stays constant in a family of Lagrangian fibrations. In particular two Lagrangian fibrations which are deformation equivalent as Lagrangian fibrations have the same polarization type.*

Proof: The proof is similar to the one of Proposition 3.3.3. Let $\phi : \mathcal{X} \to P$ be a family of Lagrangian fibrations parametrized by a complex space S. Setting $\mathcal{B} := \bigcup_{s \in S} B_s^\circ$ where as before $B_s^\circ := P_s - \Delta_s$ is the base of the Lagrangian fibration $\phi_s := \phi|_{\mathcal{X}_s} : \mathcal{X}_s \to P_s$ without the discriminant locus. Note that $\mathcal{B}$ is connected as it is P without a real codimension two subset. Set $\psi := \phi|_{\pi^{-1}(\mathcal{B})} : \phi^{-1}(\mathcal{B}) \to \mathcal{B}$ which is a holomorphic submersion.

With same argument as in Proposition 3.3.3 we get a section $A : \mathcal{B} \to R^2\psi_\star\mathbb{Z}$ such that for $t \in B_s$ the value $A(t)$ coincides with $\alpha_s(t)$, where α_s is the continuous map $\alpha_s : B_s^\circ \to \mathcal{H}_s$ as in (3.3.2) which is associated to the Lagrangian fibration ϕ_s. Let $\underline{\mathrm{d}}(A(t))$ denote the polarization type of the polarization $A(t)$ on the abelian variety $(\mathcal{X}_s)_t$ for $t \in B_s^\circ$. As $\mathcal{B}$ is connected the continuous map $\underline{\mathrm{d}}(A(\cdot)) : \mathcal{B} \to \mathbb{Z}^n$ must be constant. Since $\underline{\mathrm{d}}(\phi_s) = \underline{\mathrm{d}}(\alpha_s(t)) = \underline{\mathrm{d}}(A(t))$ for $t \in B_s^\circ$ we see that $\underline{\mathrm{d}}(\phi_s)$ is constant on S. $\square$

Proposition 3.4.4 *Let $f : X \to B$ be a Lagrangian fibration and ω a special Kähler class with respect to a smooth fiber F. Then $\underline{\mathrm{d}}(f)$ is given by the polarization type of $\omega|_F$ i.e. $\underline{\mathrm{d}}(f) = \underline{\mathrm{d}}(\omega|_F)$.*

Proof: As $\omega|_F$ is the restriction of a Kähler class it is contained in the ray G of Lemma 3.2.1. Since $\omega|_F$ is primitive it is in the image of $\alpha : B^\circ \to \mathcal{H}$ of (3.3.2) i.e. $\alpha(t) = \omega|_F$ for $F = X_t$. $\square$

Example 3.4.5 Let $f : S \to \mathbb{P}^1$ be an elliptic K3 surface and ω a special Kähler class on S with respect to a smooth fiber F. As F is an elliptic curve we have $H^2(F,\mathbb{Z}) \cong \mathbb{Z}$. Since $\omega|_F$ is primitive it is the generator of $H^2(F,\mathbb{Z})$ and so $\omega|_F = c_1(L)$ for an ample line bundle of degree $\deg(L) = 1$. By the Proposition above and [**BL03**, Thm. 3.6.3 ff.] we have

$$\underline{\mathrm{d}}(f) \;=\; \underline{\mathrm{d}}(\omega|_F) \;=\; \int_F c_1(L) \;=\; \deg(L) \;=\; 1$$

as one can identify the degree with integration of the first Chern class.

Theorem 3.4.6 *The associated Lagrangian fibrations of two marked pairs which define points in the same connected component $\mathfrak{U}^{\circ}_{\lambda^\perp}$ of the moduli of Lagrangian fibrations of $K3^{[n]}$–type or generalized Kummer type for a primitive isotropic class λ in the $K3^{[n]}$ or generalized Kummer lattice have the same polarization type.*

Proof: By Proposition 2.4.8 the associated Lagrangian fibrations are deformation equivalent and the claim follows by Proposition 3.4.3. □

A very optimistic conjecture was the following, which turned out to be false in general.

Conjecture 3.4.7 *Let $f_i : X_i \to B_i$, $i = 1, 2$, be two Lagrangian fibrations such that X_1 and X_2 are deformation equivalent. Then their polarization types coincide*

$$\underline{\mathrm{d}}(f_1) \;=\; \underline{\mathrm{d}}(f_2)\,.$$

In section 5.4 we verify this conjecture for manifolds of $\mathrm{K3}^{[n]}$–type but also show that it does not hold for the generalized Kummer type case.

3.5. A remark on Matsushita's conjecture

Let X be a projective irreducible holomorphic symplectic manifold and let L be an ample line bundle on X.

If $f : X \to B$ is a Lagrangian fibration, then L defines a polarization $c_1(L|_{X_t})$ on each smooth fiber X_t. The map $B^\circ \to \mathbb{Z}^n$, where $B^\circ \subset B$ denotes the connected open subset parametrizing the smooth fibers, which associates each $t \in B^\circ$ the type $\underline{\mathrm{d}}(L|_{X_t})$ is continuous, hence constant.

Let denote by $\underline{\lambda} = (d_1, \ldots, d_n)$ the type of those polarizations. Note that it is not clear, that $\underline{\lambda}$ is precisely the type $\underline{\mathrm{d}}(f)$ of the Lagrangian fibration $f : X \to B$, but it must be a multiple of it by Lemma 3.2.1. We get a holomorphic map

$$\begin{aligned} \phi \;:\; B^\circ &\longrightarrow \mathcal{A}_{\underline{\lambda}} \\ t &\longmapsto (X_t, c_1(L|_{X_t})) \end{aligned} \tag{3.5.1}$$

where $\mathcal{A}_{\underline{\lambda}}$ denotes the moduli space of $\underline{\lambda}$ polarized abelian varieties, cf. [**BL03**, 8.2]. The following conjecture is due to D. Matsushita.

Conjecture 3.5.2 (Matsushita) *The map $\phi : B^\circ \to \mathcal{A}_{\underline{\lambda}}$ is either generically finite on its image or constant.*

There was recently proven a weak form of this conjecture by C. Voisin and B. van Geemen, see [**vGV15**].

For a general Lagrangian fibration $f : X \to B$, in particular for a non projective one, the family of special Kähler classes $\alpha : B^\circ \to \mathcal{H}$ in (3.3.2) gives an alternative to the map (3.5.1).

Corollary 3.5.3 *For every Lagrangian fibration $f : X \to B$, there is a holomorphic map*

$$\begin{aligned} \phi \ : \ B^\circ &\longrightarrow \mathcal{A}_{\underline{\mathrm{d}}(f)} \\ t &\longmapsto (X_t, \alpha(t)) \end{aligned} \tag{3.5.4}$$

where $\mathcal{A}_{\underline{\mathrm{d}}(f)}$ denotes the moduli space of $\underline{\mathrm{d}}(f)$ polarized abelian varieties.

Note that $\underline{\mathrm{d}}(f)$ is always primitive. In this sense, we can generalize Matsushita's conjecture to every Lagrangian fibration.

CHAPTER 4

Beauville–Mukai Systems

Beauville–Mukai systems are Lagrangian fibrations defined on certain moduli spaces of sheaves on a projective holomorphic symplectic surface which are irreducible holomorphic sympletic manifolds, given by the support morphism. They are of $\mathrm{K3}^{[n]}$ or generalized Kummer type, depending on the type of surface one started with. Also examples on the O'Grady manifolds can be constructed, but one has to resolve singular moduli spaces. The advantage of those Lagrangian fibrations is, that one has a lot of methods available, coming from the theory of moduli spaces of sheaves, cf. [**HL10**].

Originally, S. Mukai [**Muk84**] noticed that moduli spaces of sheaves on symplectic surfaces carry a holomorphic symplectic form. The canonical question is when they are irreducible holomorphic symplectic, which is the case if one fixes certain numeric invariants, as the Chern classes.

This idea was used by K. O'Grady to construct his exceptional examples [**O'G99**], [**O'G03**].

Beauville–Mukai systems play an important role for the calculation of the polarization type of Lagrangian fibrations of $\mathrm{K3}^{[n]}$–type or generalized Kummer type.

In this chapter we introduce shortly the necessary background of moduli spaces of sheaves and give the precise definition of Beauville–Mukai systems. The main references are the well known book [**HL10**] and K. O'Grady's lectures notes [**O'G14b**] and [**O'G14a**] from GAeL 2014 in Trieste. But also the more specific lecture notes [**Huy15**] are good for an overview.

Further the polarization types of Beauville–Mukai systems of $\mathrm{K3}^{[n]}$ and generalized Kummer type are determined.

4.1. Support and stable sheaves

For this section, X always denotes a projective variety and F will denote a coherent sheaf on it. Our main interest is the case when X is a smooth projective surface, more precisely a projective holomorphic symplectic surface, as we consider in subsection 4.2.7.

Definition 4.1.1 (*i*) The support of a coherent sheaf F on S is as the set defined as

$$\operatorname{supp}(F) \;:=\; \{x \in X \mid F_x \neq 0\}\,.$$

(*ii*) The *annihilator* of F is the ideal defined as

$$\mathcal{A}nn(F) \coloneqq \ker(a : \mathcal{O}_S \to \mathcal{H}om_{\mathcal{O}_S}(F, F))$$

i.e. the kernel of the sheaf morphism $a : \mathcal{O}_X \to \mathcal{H}om_{\mathcal{O}_S}(F, F)$ given by $a(\sigma)(f) \coloneqq \sigma \cdot f$ for $\sigma \in \mathcal{O}_X(U)$ and $f \in F(U)$ for some open set $U \subset X$.

(*iii*) Scheme theoretically the *annihilator support* is defined as

$$\mathrm{supp}(F) \coloneqq V(\mathcal{A}nn(F))^{1}.$$

(*iv*) The *dimension* $\dim F$ of F is defined as the dimension of $\mathrm{supp}(F)$. Further F is called *pure (of dimension* $\dim F$*)* if for every nontrivial subsheaf $G \subset F$ we have $\dim G = \dim F$.

Note that we have $(\mathcal{A}nn(F))_x = \mathcal{A}nn(F_x)$, therefore we have clearly

$$(\mathcal{O}_X/\mathcal{A}nn(F))_x = 0 \text{ if and only if } F_x = 0\,,$$

hence as sets $\mathrm{supp}(\mathcal{O}_X/\mathcal{A}nn(F)) = \mathrm{supp}(F)$.

We fix a *polarization* H on X i.e. an ample Cartier divisor and we set $\mathcal{O}_X(1) \coloneqq \mathcal{O}_X(H)$ and $F(m) \coloneqq F \otimes \mathcal{O}_X(m)$. Recall that the *Hilbert polynomial (with respect to* H*)* of a coherent sheaf F is defined as $P_F(m) \coloneqq \chi(F(m))$ and we can write

$$P_F(m) \;=\; \sum_{k=0}^{\dim F} \alpha_k(F)\frac{m^k}{k!}$$

see [**HL10**, Lem. 1.2.1 ff.], where $\alpha_k(F)$ are integers. We set $\alpha_k(F) = 0$ if $k > \dim F$.

Definition 4.1.2 The *rank* of F is defined as

$$\mathrm{rk}(F) \;\coloneqq\; \frac{\alpha_{\dim X}(F)}{\alpha_{\dim X}(\mathcal{O}_X)}\,.$$

The *reduced Hilbert polynomial* of F is defined as

$$p_F(m) \;\coloneqq\; \frac{P_F(m)}{\alpha_{\dim F}(F)}\,.$$

Note that if $\dim F < \dim X$ we have clearly $\mathrm{rk}(F) = 0$. For a vector bundle i.e. a locally free sheaf the notion of the rank coincides with the classical notion, see [**HL10**, p. 11].

Example 4.1.3 Let us consider the case of a smooth projective surface S. Write r, c_1 and c_2 for the rank and Chern classes of F and by abuse of notation we also write $H = c_1(\mathcal{O}_S(1))$ for the first Chern class of the fixed ample divisor H. We can

[1] we use the notation $V(\mathcal{I}) = (Z, \mathcal{O}_Z)$ with $Z = \mathrm{supp}(\mathcal{O}_X/\mathcal{I})$ and $\mathcal{O}_Z = (\mathcal{O}_X/\mathcal{I})|_Z$ for the closed subscheme (or closed complex subspace) defined by the ideal $\mathcal{I} \subset \mathcal{O}_X$

apply Hirzebruch–Riemann–Roch to compute

$$\begin{aligned} P_F(m) &= \int_S \mathrm{ch}(F)\,\mathrm{ch}(\mathcal{O}_S(1))^m\,\mathrm{Td}_S \\ &= \int_S \left(1 + c_1 + \frac{1}{2}c_1^2 - c_2\right)\left(1 + H + \frac{1}{2}H^2\right)^m \mathrm{Td}_S \\ &= \int_S \left(1 + c_1 + \frac{1}{2}c_1^2 - c_2\right)\left(1 + mH + \frac{1}{2}m^2H^2\right)\mathrm{Td}_S \end{aligned}$$

Recall that $\mathrm{Td}_S = 1 + \frac{c_1(S)}{2} + \frac{c_1(S)^2 + c_2(S)}{12} = 1 + \frac{c_1(S)}{2} + \chi(\mathcal{O}_S)\omega$ where ω is the Poincare dual of a point. The constant term of the Hilbert polynomial is always $P_F(0) = \chi(F)$. We have

$$\begin{aligned} P_F(m) &= \frac{r}{2}(H,H)m^2 + (H, c_1 + \frac{r}{2}c_1(S))m \\ &+ \left[\frac{1}{2}(c_1, c_1 + c_1(S)) - \int_S c_2 + r\chi(\mathcal{O}_S)\right] \\ &= \frac{r}{2}(H,H)m^2 + (H, c_1 + \frac{r}{2}c_1(S))m + \chi(F) \end{aligned} \tag{4.1.4}$$

We conclude that $\alpha_2(F) = \alpha_{\dim S}(F) = r(H,H)$. If $\dim F = 2$, then the reduced Hilbert polynomial is therefore

$$p_F(m) = \frac{1}{2}m^2 + \frac{(H, c_1 + \frac{r}{2}c_1(S))}{r(H,H)}m + \frac{\chi(F)}{r(H,H)}\,.$$

We are later interested in sheaves supported on a curve i.e. $\dim F = 1$, then $\mathrm{rk}\, F = 0$ and in this case we have

$$P_F(m) = (H, c_1)m + \chi(F) \quad \text{and} \quad p_F(m) = m + \frac{\chi(F)}{(H, c_1)}\,.$$

If our surface S has trivial canonical bundle i.e. is a K3 or an abelian surface then $c_1(S) = 0$ and therefore

$$P_F(m) = \frac{r}{2}(H,H)m^2 + (H, c_1)m + \chi(F)\,.$$

Comparing (4.1.4) with $P_F(0) = \chi(F)$, we also see that in this case we have

$$\chi(F) = \frac{1}{2}(c_1, c_1) - \int_S c_2 + r\chi(\mathcal{O}_S) = \mathrm{ch}_2(F) + r\chi(\mathcal{O}_S) \tag{4.1.5}$$

where we identify $c_2 \in H^4(S, \mathbb{Z}) = \mathbb{Z}\omega$ as a number.

Definition 4.1.6 A nontrivial pure coherent sheaf F is called *H–semistable* if for all nontrivial subsheaves $E \subset F$ we have for the reduced Hilbert polynomials

$$p_E(m) \leq p_F(m) \quad \text{for all } m \gg 0\,.$$

Further F is called *H–stable* if we have strict inequality for $m \gg 0$.

Sometimes this notion of (semi)stability is also known as *Gieseker stability.*

Remark 4.1.7 A stable sheaf F is simple, that is $\mathrm{End}(F) = \mathbb{C} \cdot \mathrm{id}_F$, see [**O'G14a**, Claim 3.6] or [**HL10**, Cor. 1.2.8].

4.2. Moduli spaces of sheaves

Throughout this section, X will always denote a smooth projective variety. Also we have fixed an ample class H and we just say (semi)stable instead of H–(semi)stable if there can not be a confusion.

Recall that a *family of sheaves* $\mathcal{F}$ on X over a scheme T is a sheaf $\mathcal{F} \in \mathrm{Coh}(X \times T)$ which is flat over T. Flat over T means, that $\mathcal{F}$ is pr_T–flat where $\mathrm{pr}_T : X \times T \to T$ denotes the projection i.e. for all $(x,t) \in X \times T$, the stalk $\mathcal{F}_{(x,t)}$ is a flat $\mathcal{O}_{T,t}$–module via the morphism $\mathcal{O}_{T,t} \to \mathcal{F}_{(x,t)}$ induced by the projection pr_T.

For $t \in T$ we set $\mathcal{F}_t := (\iota_t)^\star \mathcal{F} = \mathcal{F}|_{X \times \{t\}} \in \mathrm{Coh}(X)$ where $\iota_t : X \hookrightarrow X \times T, x \mapsto (x,t)$ denotes the inclusion with respect to t.

Definition 4.2.1 For a fixed polynomial P, we define a contravariant moduli functor $\mathcal{M}_H(P) = \mathcal{M}(P) : \textbf{Schemes}\,/\mathbb{C} \to \textbf{Sets}$ via

$$\mathcal{M}(P)(T) \;:=\; \left\{ \begin{array}{c} \text{isomorphism classes of families of sheaves } \mathcal{F} \text{ on } X \text{ over } T \\ \text{such that } \mathcal{F}_t \text{ is } H\text{–semistable and } p_{\mathcal{F}_t} = P \text{ for all } t \in T \end{array} \right\} / \sim$$

where $\mathcal{F} \sim \mathcal{F}'$ if and only if $\mathcal{F} \cong \mathcal{F}' \otimes \mathrm{pr}_T^\star L$ for a line bundle $L \in \mathrm{Pic}(T)$. Further for a morphism $\varphi : T \to T'$, the associated morphism

$$\mathcal{M}(v)(\varphi) \;:\; \mathcal{M}(P)(T') \longrightarrow \mathcal{M}(P)(T)\,, \quad \mathcal{F} \longmapsto (\mathrm{id}_X \times \varphi)^\star \mathcal{F}$$

is defined via the usual pullback of sheaves by the map $\mathrm{id}_X \times \varphi : X \times T \to X \times T'$.

In the same fashion one defines a sub moduli functor $\mathcal{M}^s_H(P) : \textbf{Schemes}\,/\mathbb{C} \to \textbf{Sets}$ of $\mathcal{M}(P)$ by replacing semistable with stable.

Theorem 4.2.2 [**HL10**, Thm. 4.3.4] *The moduli functor $\mathcal{M}_H(P)$ is corepresented by a projective scheme $M_H(P)$. Further there is a open subscheme $M^s_H(P) \subset M_H(P)$ which corepresents the moduli functor $\mathcal{M}^s_H(P)$.*

Remark 4.2.3 (*i*) Corepresented by $M(P) = M_H(P)$ means, that there is a natural transformation[2] $\mathcal{M}(P) \rightsquigarrow \hom(\cdot, M(P))$ with the universal property that every other natural transformation $\mathcal{M}(P) \rightsquigarrow \hom(\cdot, N)$ for a scheme N, factorizes over an uniquely determined natural transformation $\hom(\cdot, M(P)) \rightsquigarrow \hom(\cdot, N)$.

$$\begin{tikzcd} \mathcal{M}(P) \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] & \hom(\cdot, M(P)) \arrow[dl, rightsquigarrow, dashed, "\exists!"] \\ \hom(\cdot, N) & \end{tikzcd}$$

(*ii*) The result [**HL10**, Thm. 4.3.4] also states that closed points of the scheme $M_H(P)$ are in one to one correspondence with so called *S–equivalance classes* of semistable sheaves and the closed points of $M^s_H(P)$ are in one to one correspondence with $\mathcal{S}$–equivalance classes of stable sheaves. This

[2] also called map of functors

notion is defined in terms of the *Jordan–Hölder filtration*, cf. [**HL10**, Prop. 1.5.2]: every semistable sheaf F admits a filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_l = F$$

such that the quotient F_k/F_{k-1} is stable and has reduced Hilbert polynomial p_F. Then the graded sum

$$\operatorname{gr}(F) \;\coloneqq\; \bigoplus_{k=1}^{l} F_k/F_{k-1} \tag{4.2.4}$$

does not depend of the choice of the Jordan–Hölder filtration, see [**HL10**, Prop. 1.5.2]. Two semistable sheaves F and F' are called *$\mathcal{S}$–equivalent* if $\operatorname{gr}(F) \cong \operatorname{gr}(F')$.

For any locally free sheaf F on X one has the trace map $\operatorname{Tr}_F : \mathcal{H}om(F,F) \to \mathcal{O}_X$ defined locally by the usual trace map. Since F is locally free, we have $\operatorname{Ext}^i(F,F) = H^i(\mathcal{H}om(F,F))$, therefore we get a map $\operatorname{Tr} \coloneqq H^i(\operatorname{Tr}_F) : \operatorname{Ext}^i(F,F) \to H^i(X,\mathcal{O}_X)$.

If F is a coherent sheaf, by [**Har77**, III. 6.9] we have a locally free resolution $F^\bullet : 0 \to F_n \to \cdots \to F_0 \to F \to 0$ since X is smooth. One defines a cochain map $\operatorname{Tr}_{F^\bullet} : \mathcal{H}om^\bullet(F^\bullet, F^\bullet) \to \mathcal{O}_X$ by $\operatorname{Tr}_{F^\bullet}|_{\mathcal{H}om(F_i,F_j)} = 0$ for $i \neq j$ and $\operatorname{Tr}_{F^\bullet}|_{\mathcal{H}om(F_i,F_i)} = (-1)^i \operatorname{Tr}_{F_i}$. Then the general trace map is defined as

$$\operatorname{Tr}_F \;\coloneqq\; \operatorname{Tr}_F^i \;\coloneqq\; (R^i\Gamma)(\operatorname{Tr}_{F^\bullet}) \;:\; \operatorname{Ext}^i(F,F) \longrightarrow H^i(X,\mathcal{O}_X) \tag{4.2.5}$$

where $\operatorname{Ext}^i(F,F) \cong \operatorname{Ext}^i(F^\bullet,F^\bullet)$ and $R^i\Gamma$ denotes i–th right derived functor of the section functor Γ.

Proposition 4.2.6 ([**HL10**, Cor. 4.5.2], [**Muk84**]) *Let $[F] \in M_H(P)$ be a point corresponding to the $\mathcal{S}$–equivalence class of a stable sheaf F. Then the following statements hold.*

(i) *There is a natural isomorphism*

$$\mathcal{T}_{M_H(P),[F]} \;\cong\; \operatorname{Ext}^1(F,F)$$

where $\mathcal{T}_{M_H(P),[F]}$ denotes the Zariski tangent space at $[F]$.

(ii) *If $\operatorname{Ext}^2(F,F)^0 \coloneqq \ker \operatorname{Tr}_F^2 = 0$ then $[F]$ is a smooth point of $M_H(P)$.*

4.2.7. The case of projective symplectic surfaces. We now apply the general theory to a smooth projective holomorphic symplectic surface S in sense of Definition 1.1.1. As we pointed out in section 1.1, we have $K_S = \mathcal{O}_S$, therefore by Kodaira's classification [**BHPV03**, p. 244, Table 10] of complex surfaces, S is either a K3 or an abelian surface.

We have seen in Example 4.1.3, that the Hilbert polynomial of a coherent sheaf F on a smooth projective surface only depends on the rank and the Chern classes c_1 and c_2. For considering moduli spaces it is therefore convenient to fix those numerical invariants which leads to the notion of a *Mukai vector* which is a class in the *Mukai* *lattice.*

Let $H^\bullet(S)$ denote the even cohomology ring i.e.

$$H^\bullet(S) \ := \ H^0(S,\mathbb{Z}) \oplus H^2(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z})\,.$$

It can be endowed with a weight two Hodge structure, if one defines $H^\bullet(S)^{2,0} = H^{2,0}(S)$ and $H^\bullet(S)^{1,1} = H^0(S,\mathbb{C}) \oplus H^{1,1}(S) \oplus H^4(S,\mathbb{C})$.

We can define a bilinear form on $H^\bullet(S)$ by

$$(v,w) \ := \ \int_S (v_2 \wedge w_2 - v_0 \wedge w_4 - v_4 \wedge w_o) \ = \ (v_2,w_2) - \int_S (v_0 \wedge w_4 + v_4 \wedge w_o)$$

where $(v_2,w_2) = \int_S v_2 \wedge w_2$ denotes the usual intersection form for complex surfaces on $H^2(S,\mathbb{Z})$ and $v = v_0 + v_2 + v_4$ with $v_i \in H^i(S,\mathbb{Z})$ denotes the decomposition in $H^\bullet(S)$ and similarly for w. This lattice is even, unimodular, of rank $b_2(S)+2$ i.e. 24 if S is a K3 and 8 when S is an abelian surface.

We write sometimes an element $(m, v_2, l) \in H^\bullet(S)$ as a tuple with $m, l \in \mathbb{Z}$ where we use the identification $H^0(S,\mathbb{Z}) = \mathbb{Z} = \mathbb{Z}[S]$ by taking the fundamental class $[S]$ and the identification $H^4(S,\mathbb{Z}) = \mathbb{Z} = \mathbb{Z}\omega$ where ω is the natural orientation of S coming from the complex structure or equivalently the Poincare dual of a point. Then the bilinear form reads as

$$((m,v_2,l),(m',w_2,l')) \ = \ (v_2,w_2) - ml' - lm'\,.$$

The lattice $H^\bullet(S)$ is known as the geometric *Mukai lattice* and it is isometric to the abstract *Mukai lattice*

$$\widetilde{\Lambda}_S \ := \ \begin{cases} \Lambda_{\mathrm{K3}} \oplus \mathrm{U} = E_8(-1)^{\oplus 2} \oplus \mathrm{U}^{\oplus 4} & \text{if } S \text{ is K3,} \\ \mathrm{U}^{\oplus 4} & \text{if } S \text{ is an abelian surface} \end{cases} \tag{4.2.8}$$

where $\Lambda_{\mathrm{K3}} \cong H^2(\mathrm{K3},\mathbb{Z})$ is the K3 lattice, $E_8(-1)$ the negative definite root lattice of type E_8 and U the unimodular rank two hyperbolic lattice, see (A.0.3), and for E_8 see [**BHPV03**, p. 18].

The Mukai lattice $\widetilde{\Lambda}_S$ is therefore of signature $(4, b_2(S)-2)$ i.e. $(4,20)$ if S is K3 and $(4,4)$ if S is an abelian surface.

Definition 4.2.9 The *(associated) Mukai vector* of a coherent sheaf F is defined as

$$v(F) \ := \ \mathrm{ch}(F)\sqrt{\mathrm{Td}_S} \in H^\bullet(S)\,.$$

By [**HL10**, Cor. 6.1.5] one has

$$-\,(v(F),v(G)) \ = \ \chi(F,G) \ = \ \sum_{k=0}^{2}(-1)^k \dim \mathrm{Ext}^k(F,G)\,. \tag{4.2.10}$$

Example 4.2.11 Let us consider the case of smooth projective surface S. For a coherent sheaf F with $\mathrm{rk}(F) = r$ and Chern classes c_1, c_2, we have

$$\begin{aligned} \mathrm{ch}(F) \ &= \ r + c_1 + (\tfrac{1}{2}c_1^2 - c_2) \text{ and} \\ \mathrm{Td}_S \ &= \ 1 + \frac{c_1(S)}{2} + \frac{c_1(S)^2 + c_2(S)}{12} = 1 + \frac{c_1(S)}{2} + \chi(\mathcal{O}_S)\omega \end{aligned}$$

where we have applied Noether's formula.

If S is a projective symplectic surface then $c_1(S) = 0$. In particular $c_2(S) = 12\chi(\mathcal{O}_S)\omega$ i.e. $12\chi(\mathcal{O}_S) = \int_S c_2(S)$. It is well known that $\chi(\mathcal{O}_S) = 2$ if S is K3 and $\chi(\mathcal{O}_S) = 0$ if S is an abelian surface[3]. We then have

$$\mathrm{Td}_S = \begin{cases} (1,0,2) & \text{if S is K3,} \\ (1,0,0) & \text{if S is an abelian surface.} \end{cases}$$

Hence,

$$\sqrt{\mathrm{Td}_S} = \begin{cases} (1,0,1) & \text{if S is K3,} \\ (1,0,0) & \text{if S is an abelian surface.} \end{cases}$$

We then can calculate

$$v(F) = \mathrm{ch}(F)\sqrt{\mathrm{Td}_S} = \begin{cases} (r, c_1, \frac{1}{2}c_1^2 - c_2 + r) & \text{if S is K3,} \\ (r, c_1, \frac{1}{2}c_1^2 - c_2) & \text{if S is an abelian surface.} \end{cases}$$

If we use (4.1.5) and denote by $\chi = \chi(F)$ the holomorphic Euler characteristic of F, then we can rewrite this as

$$v(F) = \begin{cases} (r, c_1, \chi - r) & \text{if S is K3,} \\ (r, c_1, \chi) & \text{if S is an abelian surface.} \end{cases}$$

As we have pointed out, the Hilbert polynomial of a coherent sheaf F on a smooth projective surface only depends on the rank and the Chern classes c_1 and c_2. Considering moduli spaces we should therefore fix a class $v \in H^\bullet(S)$.

Definition 4.2.12 An *(abstract) Mukai vector* is a class $v = (r, c, s)$ in $H^\bullet(S)$ such that $r \geq 0$, $c \in H^{1,1}(S, \mathbb{Z})$ and c is effective if $r = 0$.

With this definition the geometric Mukai vector of a coherent sheaf is also an abstract Mukai vector.

Now fix a Mukai vector $v \in H^\bullet(S)$, the reduced Hilbert polynomial of each coherent sheaf with this Mukai vector is determined by it, call it P_v. If $\mathcal{F} \in \mathrm{Coh}(X \times S)$ is a family of sheaves with S connected, then $v(\mathcal{F}_s)$ is constant.

Therefore one can define a moduli functor

$$\mathcal{M}_H(v)(T) := \{\mathcal{F} \in \mathcal{M}_H(P_v) \mid v(\mathcal{F}_t) = v \text{ for all } t \in T\}$$

and $\mathcal{M}_H^s(v)$ similar as in Definition 4.2.1.

Definition 4.2.13 Denote by $M_H^s(v) \subset M_H(v)$ the corresponding moduli spaces whose existence is implied by Theorem 4.2.2, parametrizing $\mathcal{S}$–equivalance classes of (semi)stable sheaves with Mukai vector v.

[3]One has $h^{p,q}(T) = \binom{n}{p}\binom{n}{q}$ for a complex torus T of dimension n.

Let $[F] \in M_H^s(v)$ be an $\mathcal{S}$–equivalence class of a stable sheaf. As $K_S = \mathcal{O}_S$ we have by Serre duality [**Huy06**, p. 67] $\mathrm{Ext}^2(F,F) = \mathrm{Ext}^0(F,F)^\vee = \mathrm{End}(F)^\vee$. By Remark 4.1.7 F is simple i.e. $\mathrm{End}(F) \cong \mathbb{C}$, therefore $\mathrm{Ext}^2(F,F) \cong \mathbb{C}$. As $K_S = \mathcal{O}_S$ i.e. $H^2(S,\mathcal{O}_S) \cong \mathbb{C}$ the trace map $\mathrm{Tr}_F^2 : \mathrm{Ext}^2(F,F) \to H^2(S,\mathcal{O}_S)$ must be an isomorphism, as it is nontrivial. Therefore $\mathrm{Ext}^2(F,F)^0 = 0$ and Proposition 4.2.6 applies i.e. F is a smooth point of $M_H^s(v)$ and

$$\dim M_H(v) \;=\; \dim M_H^s(v) \;=\; \dim \mathcal{T}_{M_H(v),[F]} \;=\; \dim \mathrm{Ext}^1(F,F) \;=\; 2+(v,v)$$

where we have used (4.2.10). We conclude the following from the more general statements above.

Proposition 4.2.14 *The moduli space $M_H^s(v)$ of stable sheaves with Mukai vector v is either empty or a smooth quasi–projective variety of dimension $2+(v,v)$.*

The holomorphic symplectic form on S inherits a holomorphic symplectic form on $M_H^2(v)$. For writing it explicitly down, we briefly recall the Yoneda product $\cup : \mathrm{Ext}^1(F,F) \otimes \mathrm{Ext}^1(F,F) \to \mathrm{Ext}^2(F,F)$ for a coherent sheaf F, cf. [**HL10**, 10.1]. The simplest way to define it, is in terms of extensions: for $E = [0 \to F \to G \to F \to 0]$ and $E' = [0 \to F \to G' \to F \to 0]$ in $\mathrm{Ext}^1(F,F)$ one sets $E \cup E' := [0 \to F \to G \to G' \to F \to 0] \in \mathrm{Ext}^2(F,F)$. For F locally free the Yoneda product can be obtained by

$$H^1(\mathcal{E}nd(F)) \otimes H^1(\mathcal{E}nd(F)) \longrightarrow H^2(\mathcal{E}nd(F) \otimes \mathcal{E}nd(F)) \longrightarrow H^2(\mathcal{E}nd(F))$$

where the first map is the natural pairing (cf. [**Voi02**, 5.3.2]), the second map is induced by composition in $\mathcal{E}nd(F)$ and we use $\mathrm{Ext}^i(F,F) \cong H^i(\mathcal{E}nd(F))$.

Theorem 4.2.15 (Mukai, [**Muk84**], [**HL10**, Thm. 10.4.3]) *The moduli space $M_H^s(v)$ of stable sheaves carries a holomorphic symplectic form σ, which is for $\alpha, \beta \in \mathcal{T}_{M_H(v),[F]} = \mathrm{Ext}^1(F,F)$ pointwise given by*

$$\sigma_{[F]}(\alpha,\beta) \;=\; \int_S \sigma_S \wedge \mathrm{Tr}_F^2(\alpha \cup \beta)$$

where σ_S denotes a fixed holomorphic sympletic form on S.

The natural question is, when the compactification $M_H(v)$ of $M_H^s(v)$ is an irreducible holomorphic symplectic manifold. For that, we need to choose a specific polarization H on S, called *v–generic*, for which there are no strictly H–semistable sheaves (i.e. semistable but not stable).

Definition 4.2.16 ([**Zow10**, 1.4.1], [**O'G14a**, 3.16], [**HL10**, 4.C]) Let $\mathrm{Amp}(S) \subset \mathrm{NS}(S)$ denote the ample cone of S and $\mathrm{Amp}(S)_\mathbb{R} = \mathrm{Amp}(S) \otimes \mathbb{R}$. A *wall* in $\mathrm{Amp}(S)_\mathbb{R}$ is a hyperplane defined by $W_D := D^\perp \cap \mathrm{Amp}(S)_\mathbb{R}$ where D is a divisor with strictly negative self intersection $(D,D) < 0$.

Now fix a Mukai vector $v = (r, c, s) \in H^\bullet(S)$. A *$v$-wall* W_D is a wall with $D \in \mathrm{Num}(S) = \mathrm{NS}(S)/\text{torsion}$ and $\frac{r^2}{4}\Delta_v < (D, D)$ where[4] $\Delta_v := 2rs - (r-1)(c, c)$. A polarization H is called *v–generic* if it is not contained in a v–wall.

Proposition 4.2.17 [**HL10**, Thm. 4.C.3] *Let v denote a primitive Mukai vector and H a v–generic polarization. Then every H–semistable sheaf is H–stable, with other words $M^s_H(v) = M_H(v)$. In particular $M_H(v)$ is a projective holomorphic symplectic manifold.*

4.2.18. Mukai's homomorhpism. Let S be a projective holomorphic symplectic surface. For a primitive Mukai vector $v \in H^\bullet(S)$ and H a v–generic polarization, we have Mukai's homomorphism of Hodge structures

$$\Theta_v \;:\; v^\perp \longrightarrow H^2(M_H(v), \mathbb{Q})$$

which can be defined as follows. Choose a quasi–universal family of sheaves $\mathcal{E}$ on S of simplitude $\rho \in \mathbb{N}$, cf. [**Muk87**, Thm. A.5]. That is a family of sheaves $\mathcal{E} \in \mathrm{Coh}(S \times M_H(v))$ on S parametrized by $M_H(v)$ (in particular, $\mathcal{E}$ is flat over $M_H(v)$) and for every class $F \in M_H(v)$ one has $\mathcal{E}_{[F]} = \mathcal{E}|_{S\times\{F\}} \cong F^{\oplus\rho}$. Then set

$$\Theta_v(x) \;:=\; \frac{1}{\rho}\left[(\mathrm{pr}_{M_H(v)})_!\left((\mathrm{ch}(\mathcal{E})(\mathrm{pr}_S)^\star(\sqrt{\mathrm{Td}(S)}x^\vee)\right)\right]_2$$

where $x^\vee = -x_0 + x_2 + x_4$ for $x = x_0 + x_2 + x_4$ and $[\cdot]_2$ denotes the part in $H^2(S \times M_H(v), \mathbb{Z})$. For details see [**Yos01**, 1.2], [**O'G97**], [**Muk87**] and [**Muk84**].

4.2.19. The K3 case. The following Theorem is a result by many authors, the most general form is due to K. Yoshioka.

Theorem 4.2.20 ([**O'G97**], [**HL10**, Thm. 6.2.5], [**Yos01**]) *Let S be a $K3$ surface, $v = (r, c, s) \in H^\bullet(S)$ be a primitive Mukai vector as in* Definition 4.2.12 *with $(v, v) \geq 0$, $(r, s) \neq (0, 0)$ and H a v–generic polarization. Then the moduli space $M_H(v)$ is an irreducible holomorphic symplectic manifold of $K3^{[n]}$–type where $2n = 2 + (v, v)$. Further if $(v, v) \geq 2$, then Mukai's homomorphism of Hodge structures*

$$\Theta_v \;:\; v^\perp \longrightarrow H^2(M_H(v), \mathbb{Z})$$

is integral and an isometry.

4.2.21. The torus case. We assume now that S is an abelian surface. In this case, we need a small modification to obtain an irreducible holomorphic symplectic manifold.

If $v = (r, c, s)$ is a primitive Mukai vector on S with $(v, v) \geq 6$, $(r, s) \neq 0$ and H a v–generic polarization, then by [**Yos01**, Thm. 0.1] the Albanese torus of $M_H(v)$ is $S \times S^\vee$. After choosing a reference point, we can therefore write

$$\mathrm{Alb}_v \;:=\; \mathrm{Alb}_{M_H(v)} \;:\; M_H(v) \longrightarrow S \times S^\vee$$

[4] Δ_v is the *discriminant* of a sheaf F with $v(F) = v$.

and consider the fiber $K_H(v) := \mathrm{Alb}_v^{-1}(0,0)$. Then $\dim K_H(v) = \dim M_H(v) - 4 = (v,v) - 2$. By composing Mukai's homomorphism of Hodge structures with the restriction mapping $r : H^2(M_H(v),\mathbb{Z}) \to H^2(K_H(v),\mathbb{Z})$ we obtain an morphism

$$\Theta_v \ : \ v^\perp \longrightarrow H^2(M_H(v),\mathbb{Q}) \longrightarrow H^2(K_H(v),\mathbb{Q})$$

of Hodge structures which we also denote by Θ_v and call Mukai's homomorphism by abuse of notation.

Theorem 4.2.22 [**Yos01**, Thm. 0.2] *Let S be an abelian surface, $v = (r,c,s) \in H^\bullet(S)$ be a primitive Mukai vector as in* Definition 4.2.12 *with $(v,v) \geq 6$, $(r,s) \neq (0,0)$ and H a v–generic polarization. Then the fiber $K_H(v)$ of the Albanese map is an irreducible holomorphic symplectic manifold of generalized Kummer n–type where $2n = (v,v) - 2$. Further Mukai's homomorphism of Hodge structures*

$$\Theta_v \ : \ v^\perp \longrightarrow H^2(M_H(v),\mathbb{Z}) \longrightarrow H^2(K_H(v),\mathbb{Z})$$

is integral and an isometry.

4.3. An orbit of primitive isometric embeddings

The main ingredient for the construction of a monodromy invariant for isotropic classes in the second cohomology of a generalized Kummer manifold is a monodromy invariant orbit of primitive isometric embeddings of the generalized Kummer type lattice into the Mukai lattice.

Let X be a $\mathrm{K3}^{[n]}$ or generalized Kummer type manifold and set $\Lambda := H^2(X,\mathbb{Z})$. Let $\widetilde{\Lambda}$ denote the associated Mukai lattice (4.2.8) i.e. we use now the notation that

$$\widetilde{\Lambda} \ := \begin{cases} \Lambda_{\mathrm{K3}} \oplus \mathrm{U} = E_8(-1)^{\oplus 2} \oplus U^{\oplus 4} & \text{if } \Lambda \text{ is the } \mathrm{K3}^{[n]} \text{ lattice,} \\ \mathrm{U}^{\oplus 4} & \text{if } \Lambda \text{ is the generalized Kummer lattice.} \end{cases}$$

The group of isometries $\mathrm{O}(\widetilde{\Lambda})$ of the Mukai lattice and $\mathrm{O}(\Lambda)$ acts on the set $\mathrm{O}(\Lambda,\widetilde{\Lambda})$ of primitive isometric embeddings $\iota : \Lambda \hookrightarrow \widetilde{\Lambda}$ of the lattice Λ into $\widetilde{\Lambda}$ by composition i.e. for $g \in \mathrm{O}(\Lambda)$ and $\tilde{g} \in \mathrm{O}(\widetilde{\Lambda})$ one sets $g \cdot \iota := \iota \circ g$ and $\tilde{g} \cdot \iota := \tilde{g} \circ \iota$.

Definition 4.3.1 Let $\iota \in \mathrm{O}(\Lambda,\widetilde{\Lambda})$ be a primitive isometric embedding. An element $g \in \mathrm{O}(\Lambda)$ *leaves the* $\mathrm{O}(\widetilde{\Lambda})$*–orbit* $[\iota] = \mathrm{O}(\widetilde{\Lambda})\iota$ *invariant* if $g \cdot [\iota] := [\iota \circ g] = [\iota]$ i.e. if there exists $\tilde{g} \in \mathrm{O}(\widetilde{\Lambda})$ such that $\tilde{g} \circ \iota = \iota \circ g$. The orbit is called *monodromy invariant* if $\mathrm{Mon}^2(X) \cdot [\iota] = [\iota]$ i.e. all elements in $\mathrm{Mon}^2(X)$ leave the orbit $[\iota]$ invariant.

Remark 4.3.2 Let $\iota : \Lambda \hookrightarrow \widetilde{\Lambda}$ denote a primitive isometric embedding.

(*i*) If X is a generalized Kummer type manifold then $\iota(\Lambda)^\perp = \langle v \rangle$ is of rank 1 since the Mukai lattice if of rank 8 and the Kummer type lattice is of rank 7. An isometry $\tilde{g} \in \mathrm{O}(\widetilde{\Lambda})$ with $\iota \circ g = \tilde{g} \circ \iota$ necessarily satisfies $\tilde{g}(\iota(\Lambda)) = \iota(\Lambda)$ and $\tilde{g}(v) = \pm v$, otherwise $\tilde{g}$ cannot be an isometry.

(*ii*) If X of $\mathrm{K3}^{[n]}$–type, then with the same argument, also $\iota(\Lambda)^\perp = \langle v\rangle$ is of rank 1 since the Mukai lattice is of rank 24 and the $\mathrm{K3}^{[n]}$ lattice is of rank 23. Again we have $\tilde{g}(\iota(\Lambda)) = \iota(\Lambda)$ and $\tilde{g}(v) = \pm v$.

E. Markman has shown the following for the $\mathrm{K3}^{[n]}$–type case.

Theorem 4.3.3 (Markman, [**Mar11**, Cor. 9.5], [**Mar10**, Thm. 1.10]) *Let X be a $K3^{[n]}$–type manifold, $n \geq 2$. Then there exists a canonical*[5] *monodromy invariant $\mathrm{O}(\widetilde{\Lambda})$–orbit ι_X of primitive isometric embeddings $\Lambda = H^2(X,\mathbb{Z}) \hookrightarrow \widetilde{\Lambda}$.*

The following Lemma is a special case of [**Nik80**, Cor. 1.5.2].

Lemma 4.3.4 *Let Λ be the generalized Kummer or $K3^{[n]}$ lattice. Write $\Lambda = w^\perp \subset \widetilde{\Lambda}$ with w primitive (cf.* Remark 4.3.2*). An isometry $g \in \mathrm{O}(\Lambda)$ can be extended to an isometry $\tilde{g} \in \mathrm{O}(\widetilde{\Lambda})$ if and only if g acts as ± 1 on the discriminant $\Lambda^\vee/\Lambda$.*

Proof: By [**Nik80**, Cor. 1.5.2] we can extend g to such a $\tilde{g}$ if and only if we have an isometry $\varphi : \Lambda^\perp \to \Lambda^\perp$ with an additional property. Since $\Lambda^\perp = \langle w\rangle$ the only two isometries are $\varphi = \pm 1$. Following the exposition in [**Nik80**, 5. ff.], the additionally property for $\varphi = \pm 1$ means that g acts on $\Lambda^\vee/\Lambda$ as ± 1. □

Corollary 4.3.5 *Let $\Lambda = w^\perp \subset \widetilde{\Lambda}$ be as in the Lemma above and let denote $[\iota] = \mathrm{O}(\widetilde{\Lambda})\iota$ an arbitrary $\mathrm{O}(\widetilde{\Lambda})$–orbit of primitive isometric embeddings $\Lambda \hookrightarrow \widetilde{\Lambda}$. Then the sub group $\mathcal{W}(\Lambda) \subset \mathrm{O}^+(\Lambda)$ defined in* Definition 1.5.7 *is equal to the sub group of all $g \in \mathrm{O}^+(\Lambda)$ leaving the orbit $[\iota] = \mathrm{O}(\widetilde{\Lambda})\iota$ invariant, i.e. there exists $\tilde{g}$ such that $\iota \circ g = \tilde{g} \circ \iota$.*

Proof: An element $g \in \mathrm{O}^+(\Lambda)$ leaves $\mathrm{O}(\widetilde{\Lambda})\iota$ invariant if and only if it acts by ± 1 on the discriminant $\Lambda^\vee/\Lambda$ by Lemma 4.3.4. □

In other words, $\mathcal{W}(\Lambda) = \mathrm{Stab}([\iota])$ is equal to the stabilizer of $[\iota]$ with respect to the action of $\mathrm{O}^+(\Lambda)$ on the set of $\mathrm{O}(\widetilde{\Lambda})$–orbits of primitive isometric embeddings $\mathrm{O}(\Lambda, \widetilde{\Lambda})$. The Corollary immediately implies the following.

Corollary 4.3.6 [**Mar11**, Cor. 9.5] *Let X be a $K3^{[n]}$–type manifold, $\Lambda := H^2(X,\mathbb{Z})$ and let ι_X denote a monodromy invariant orbit of primitive isometric embeddings, cf.* Theorem 4.3.3*. Then $\mathrm{Mon}^2(X) = \mathrm{Stab}(\iota_X)$ is equal to the stabilizer of ι_X with respect to the action of $\mathrm{O}^+(\Lambda)$ on the set of $\mathrm{O}(\widetilde{\Lambda})$–orbits of primitive isometric embeddings $\mathrm{O}(\Lambda, \widetilde{\Lambda})$.*

Proof: We have $\mathrm{Mon}^2(X) = \mathcal{W}(X)$ by Theorem 1.5.11 and $\mathcal{W}(X) = \mathrm{Stab}(\iota_X)$ by Corollary 4.3.5. □

4.3.7. The generalized Kummer type case. With the knowledge of the monodromy group of a generalized Kummer manifold, see Theorem 1.5.12, one can

[5] See proof of Theorem 4.3.8 for the meaning of *canonical*.

construct an analogue of the monodromy invariant $\mathrm{O}(\widetilde{\Lambda})$–orbit as in Theorem 4.3.3.

Let S be an abelian surface and choose a positive and primitive Mukai vector $v = (r, c, s)$ with $c \in \mathrm{NS}(S)$ and $(v, v) \geq 6$ together with a v–generic polarization H. By Theorem 4.2.22 we know that the fiber $K_H(v) := \mathrm{Alb}_v^{-1}(0,0)$ of the Albanese map $\mathrm{Alb}_v : M_H(v) \longrightarrow S \times S^\vee$ is a generalized Kummer type manifold and $\dim K_H(v) = (v, v) - 2 =: 2n$. Further we have Mukai's homomorphism

$$\Theta_v \ : \ v^\perp \longrightarrow H^2(M_H(v), \mathbb{Z}) \longrightarrow H^2(K_H(v), \mathbb{Z})$$

which is an isometry and respects the Hodge structures.

Theorem 4.3.8 *Let X be a manifold of generalized Kummer type of dimension $2n \geq 4$. Then there exists a monodromy invariant $\mathrm{O}(\widetilde{\Lambda})$–orbit ι_X of primitive isometric embeddings $\Lambda = H^2(X, \mathbb{Z}) \hookrightarrow \widetilde{\Lambda}$ into the Mukai lattice.*

Proof: Let $K_H(v)$ denote the manifold of generalized Kummer type described above such that $\dim X = \dim K_H(v)$. Fix an isometry $\varphi : H^\bullet(S) \to \widetilde{\Lambda}$ and let $P : H^2(X, \mathbb{Z}) \to H^2(K_H(v), \mathbb{Z})$ be a parallel transport operator. Denote by ι the primitive isometric embedding

$$H^2(X, \mathbb{Z}) \xrightarrow{P} H^2(K_H(v), \mathbb{Z}) \xrightarrow{\Theta_v^{-1}} v^\perp \xrightarrow{\varphi} \widetilde{\Lambda} \ .$$

Set $\iota_X := \mathrm{O}(\widetilde{\Lambda})\iota$. Let $g \in \mathrm{Mon}^2(X)$ denote a monodromy operator. By Theorem 1.5.12 g acts on $H^2(X, \mathbb{Z})^\vee / H^2(X, \mathbb{Z})$ as $\pm \mathrm{id}$. By Lemma 4.3.4 g can be extended to an isometry $\tilde{g}$ of $\widetilde{\Lambda}$ such that $\iota \circ g = \tilde{g} \circ \iota$, i.e. the orbit ι_X is monodromy invariant.

The orbit ι_X is *canonical* in the following sense. We have made a choice of moduli spaces $K_H(v) \subset M_H(v)$ of sheaves on an abelian surface S and therefore of Mukai's homomorphism $\Theta_v : v^\perp \to H^2(M_H(v), \mathbb{Z}) \to H^2(K_H(v), \mathbb{Z})$. It might be, that a different choice of moduli spaces and therefore of a different Mukai homomorphism could lead to another orbit of primitive isometric embeddings. With *canonical* we mean that we always end up with the same orbit.

This follows from K. Yoshioka's method of proof of Theorem 4.2.22, see [**Yos01**, 4.3., Prop. 4.12., Proof of Thm. 0.1 and 0.2]. If we choose another irreducible holomorphic symplectic moduli space of dimension $\dim X$, then it is deformation equivalent to $K_H(v)$ and Yoshioka's proof for this statement uses deformations of moduli spaces of sheaves over families of surfaces [**Yos01**, Lem. 2.3], and Fourier–Mukai transforms for which the Mukai homomorphism varies continuously, see [**Yos01**, 2.2., Proof of Prop. 2.4.]. Therefore the $\mathrm{O}(\widetilde{\Lambda})$–orbit does not change. □

4.4. Beauville–Mukai systems and their polarization type

Let S be a projective holomorphic symplectic surface, $v = (r, c, s)$ a Mukai vector and H an v–generic polarization. For instance, one can take the requirements as in Theorem 4.2.20 or Theorem 4.2.22. We want to consider certain Lagrangian fibrations defined on $X = M_H(v)$ or $X = K_H(v)$, respectively, which are obtained by the support morphism, called *Beauville–Mukai systems*, which were first introduced by A. Beauville [**Bea99**], see also Remark 4.4.13.

First note, that on a smooth curve, the notions of a pure, a torsion free and a locally free sheaf are equivalent, see [**HL10**, Def. 1.1.4] and [**HL10**, Ex. 1.1.16].

Example 4.4.1 Let S be a projective K3 surface, and F a torsion free sheaf on S such that $C \coloneqq \mathrm{supp}(F)$ is a smooth irreducible curve of genus g with $\mathrm{rk}(F|_C) = 1$. Then $\mathrm{rk}(F) = 0$ and $c_1(F) = c_1(C)$. We have $H^i(C, F|_C) = H^i(S, F)$ (see this for instance with Čech cohomology), hence $\chi(F|_C) = \chi(F)$. The Riemann–Roch for curves gives

$$v(F) \;=\; (0, c_1(F), \chi(F) - \mathrm{rk}(F)) \;=\; (0, C, 1 - g + d)$$

where d denotes the degree of the restriction of F to C.

If we take a stable sheaf F on S, we want the support $\mathrm{supp}(F)$ to be an element of a lower dimensional space. The example suggests as a canonical candidate, the linear system $|D|$ (or more precisely the dual linear system $|D|^\star$) coming from $c \in H^{1,1}(S)$ i.e. D is a divisor with $c_1(D) = c$. But then, $\mathrm{supp}(F)$ is of dimension one, hence $\mathrm{rk}(F) = 0$. Therefore we should set $r = 0$ and consider Mukai vectors of the form

$$v \;=\; (0, c_1(D), s) \;=\; (0, [D], s)$$

for some effective divisor D on S. In this case we have $\dim M_H(v) = (D, D) + 2$ and $\dim K_H(v) = (D, D) - 2$.

4.4.2. Fitting support. There is also a different natural scheme structure on the support of a sheaf, called the *fitting structure*.

Let F be a coherent sheaf on S. By definition S can be covered by open sets $U \subset S$ such that there is an exact squence

$$\mathcal{O}_U^q \xrightarrow{A} \mathcal{O}_U^p \longrightarrow \mathcal{F} \longrightarrow 0\,.$$

For every open subset $V \subset U$ one can define an ideal sheaf $\mathscr{F}itt_V \subset \mathcal{O}_S(V)$ which is generated by all p–minors of A, that is, the determinants of $p \times p$ submatrices of A and is independent of the chosen exact sequence above, see [**Eis95**].

Definition 4.4.3 Let F be a coherent sheaf on S.

(i) The 0–*th fitting ideal* $\mathscr{F}itt(F) = \mathscr{F}itt_0(F)$ of F is the ideal sheaf of $\mathcal{O}_S$ defined by gluing all locally defined ideals $\mathscr{F}itt_V$ together.

(ii) The *fitting support* of F is the subspace of S defined as

$$\mathrm{supp}(F) \coloneqq V(\mathscr{F}itt(F))\,.$$

For the general definition of fitting ideals see [**Eis95**] and [**Tei76**]. Note that we have $F_x = 0$ iff A_x is surjective iff a $p \times p$ minor of A_x is a unit iff $\mathscr{F}itt(F_x) = \mathcal{O}_{S,x}$. Therefore as topological spaces, annihilator and fitting support are the same. Of course one can define fitting ideals on arbitrary complex spaces – we have not used that we are on a surface.

The advantage of the fitting support is that the fundamental class $[\mathrm{supp}(F)]$ is $c_1(F)$ and that it behaves well in families, see the following Lemmas.

Lemma 4.4.4 [**Sac13**, Lem. 1.1.6] *Let $\mathcal{F}$ be a family of pure sheaves of dimension one on a smooth projective surface S over T. There exists a subscheme $\Gamma \subset S \times T$ such that $\Gamma_t = \mathrm{supp}(\mathcal{F}_t)$ for all $t \in T$, where we mean the fitting support.*

Recall that the *determinant* of a locally free sheaf F of rank r is defined as $\det(F) \coloneqq \bigwedge^r F$. For F coherent, one uses a locally free resolution $0 \to F_l \to \cdots F_0 \to F \to 0$ and sets $\det(F) \coloneqq \bigotimes_i \det(F)^{(-1)^i}$, cf. [**HL10**, 1.1.17].

Remark 4.4.5 We consider the case, when F is a pure sheaf of dimension one on a smooth projective surface.

(i) F admits a locally free resolution

$$0 \longrightarrow F_1 \xrightarrow{A} F_0 \longrightarrow F \longrightarrow 0$$

of length one, see [**HL10**, p. 4 ff.]. If $x \notin \mathrm{supp}(F)$, then $F_x \neq 0$ i.e. $(F_1)_x \cong (F_0)_x$, hence F_1 and F_0 have the same rank, as they are locally free.

(ii) In this case $\det(F) = \det(F_0) \otimes \det(F_1)^{-1}$.

(iii) The morphism A induces a morphism $\det(F_1) \to \det(F_0)$, given by $f_1 \wedge \ldots \wedge f_r \mapsto A(f_1) \wedge \ldots \wedge A(f_r)$. Note that you can rewrite this locally as $s \mapsto \det(A)s$, where $\det(A)$ is the determinant of the matrix A. In particular the fitting ideal $\mathscr{F}itt_0(F)$ is generated by $\det(A)$. Furthermore, $\mathscr{F}itt_0(F)_x \neq \mathcal{O}_{S,x}$ iff $\det(A_x)$ is not invertible which is the case iff $\det(A)(x) = 0$. Consequently the fitting support $\mathrm{supp}(F)$ is given by a single equation and is therefore a divisor in S.

The next crucial property of the fitting support in our situation is the following Lemma.

Lemma 4.4.6 *For a pure sheaf F of dimension one on a smooth projective surface one has*

$$\mathcal{O}_S(\mathrm{supp}\, F) \;=\; \det(F)\,,$$

if $\mathrm{supp}(F)$ *is endowed with the fitting scheme structure (and is therefore a divisor, cf.* Remark 4.4.5 (iii)*). In particular $c_1(\mathrm{supp}(F)) = c_1(F)$.*

Proof: Choose a locally free resolution

$$0 \longrightarrow F_1 \xrightarrow{A} F_0 \longrightarrow F \longrightarrow 0$$

as in the Remark 4.4.5 (i) above. Let $D = \mathrm{supp}(F)$ denote the divisor given by the support of F. The associated sequence of the determinants with restricted $\det(F)|_D$

$$0 \longrightarrow \det(F_1) \xrightarrow{A} \det(F_0) \longrightarrow \det(F)|_D \longrightarrow 0$$

is exact. By tensoring with $\det(F_0)^{-1}$ we obtain

$$0 \longrightarrow \det(F_0)^{-1} \otimes \det(F_1) \xrightarrow{A} \mathcal{O}_S \longrightarrow \mathcal{O}_D \longrightarrow 0\,.$$

We have $\det(F_0)^{-1} \otimes \det(F_1) = \det(F)^{-1}$ by definition, but the sequence above must be the structure sequence of D i.e. $\det(F)^{-1} = \mathcal{O}_S(-D)$, that is $\det(F) = \mathcal{O}_S(D)$. The second statement follows from the fact, that $c_1(F) = c_1(\det(F))$ for any coherent sheaf F. □

Following [**Pot93**, p. 24] we can define the following morphism.

Definition 4.4.7 Let $v = (0, c_1(D), s)$ be a Mukai vector where D is an effective divisor on S and let $\{D\}$ denote the irreducible component of the Hilbert scheme of S which parameterizes subschemes with fundamental class $[D] = c_1(D)$ and contains D. We have a holomorphic map, the *support morphism*

$$\begin{aligned} \pi \;=\; \pi_v \;:\; M_H(v) &\longrightarrow \{D\} \\ F &\longmapsto \operatorname{supp}(F) \end{aligned}$$

where $\operatorname{supp}(F)$ is endowed with the fitting scheme structure. If $C \in \{D\}$ is a smooth curve, then the fiber $\pi_v^{-1}(C) \cong \operatorname{Jac}^d(C)$ is the Jacobian of C of some degree d by construction, where d is determined by v, see Remark 4.4.10.

Remark 4.4.8 (*i*) Indeed, the support map is a morphism by Lemma 4.4.4: for $F \in M_H(v)$ there is a neighbourhood U of F and a universal family of sheaves $\mathcal{F} \in \operatorname{Coh}(S \times U)$ with $[\mathcal{F}_u] = u$ for all $u \in U$, hence $\pi|_U(u) = \operatorname{supp}(\mathcal{F}_u) = R_u$ which is clearly regular, as $\operatorname{pr}_U : R \to U$ defines a deformation.

(*ii*) In the setting of Definition 4.4.7 we have the *determinant morphism*

$$\begin{aligned} \det \;:\; M_H(v) &\longrightarrow \operatorname{Pic}(S) \\ F &\longmapsto \det(F)\,. \end{aligned}$$

If S is a K3 surface, then this is just the constant map with value $\mathcal{O}_S(D)$. If S is an abelian surface, then we clearly have $\det^{-1}(\mathcal{O}_S(D)) = \pi^{-1}(|D|)$ for the linear system $|D| \subset \{D\}$, cf. Lemma 4.4.6.

Remark 4.4.9 If D is a *big and nef*[6] divisor on S, by the Kodaira–Ramanujam theorem [**Huy15**, 2. Prop. 1.8] we have $H^i(S, D \otimes K_S) = 0$ for $i > 0$. As $K_S = \mathcal{O}_S$, we compute

$$h^0(S, D) \;=\; \chi(\mathcal{O}_S) + \frac{1}{2}(D, D) \;=\; \begin{cases} \frac{1}{2}(D,D) + 2 & \text{if } S \text{ is K3,} \\ \frac{1}{2}(D,D) & \text{if } S \text{ is an abelian surface,} \end{cases}$$

by Riemann–Roch for surfaces. Therefore we should choose D as a big and nef divisor to have the right expected dimension for $|D| \cong \mathbb{P}^n$ with $n = h^0(S, D) - 1$. On a abelian surface a line bundle L is ample iff $h^0(S, L) \neq 0$, $h^1(S, L) = 0$ and $(L, L) > 0$ by [**BL03**, Thm. 3.4.5, Prop. 4.5.2]. Therefore on an abelian surface a divisor D is big and nef if and only if D is ample.

[6] A divisor on a surface is called *big and nef* if $(D, D) > 0$ and D is nef.

Remark 4.4.10 We use the same notation as in Definition 4.4.7. The degree d of the Jacobian can be determined in the following way. Let $g = 1 + \frac{1}{2}(D,D)$ denote the arithmetic genus of D, in particular $g(C) = g$. For $F \in \pi_v^{-1}(C)$ we have $(0, c_1(D), s) = v(F) = (0, c_1(F), \chi(F))$. As $\chi(F) = \chi(F|_C)$ (use Čech cohomology) and $F|_C$ is a line bundle, we can use the Riemann–Roch on C to see

$$s \; = \; 1 - g + \deg(F|_C)\,.$$

Hence, $d = \deg(F|_C) = s + g - 1$. If D is big and nef and we set $n \coloneqq h^0(S,D) - 1$, then we get $g = n$ if S is a K3 and $g = n+2$ if S is an abelian surface. Hence,

$$d \; = \; \begin{cases} s + n - 1 & \text{if S is K3 and} \\ s + n + 1 & \text{if S is an abelian surface.} \end{cases}$$

4.4.11. Beauville–Mukai systems of K3$^{[n]}$–type. Let S be a projective K3 surface and v be a primitive Mukai vector on S of the form $v = (0, c_1(D), s)$ where D is a big and nef divisor on S. We set $2n \coloneqq (D,D) + 2$. By the dicussion above we have $h^0(S,D) = \frac{1}{2}(D,D) + 2 = n + 1$. Choose a v–generic ample class H on S, hence $M_H(v)$ is an irreducible holomorphic symplectic manifold by Theorem 4.2.20. It comes with a natural Lagrangian fibration in the following way.

The space $M_H(v)$ parametrizes sheaves F on S with $v(F) = v$ i.e. F is of rank zero with Chern class $c_1(F) = c_1(D)$. Since $H^1(S, \mathcal{O}_S) = 0$, we have that $|D| = \{D\}$. In particular F is supported on a divisor which is an element of $|D| \cong \mathbb{P}^n$. By Definition 4.4.7 we can use the support morphism

$$\pi : M_H(v) \longrightarrow |D|\,, \quad F \longmapsto \operatorname{supp} F$$

to obtain a holomorphic map $M_H(v) \to |D|$. Since $M_H(v)$ is irreducible holomorphic symplectic π is a Lagrangian fibration by Matsushita's Theorem 2.1.3. If C is a smooth curve and an element of $|D|$ then the fiber $\pi^{-1}(C)$ is the Jacobian of the curve C by construction.

Definition 4.4.12 Let S be a projective K3 surface. Fix a big and nef divisor D on S, a primitive Mukai vector of the form $v = (0, c_1(D), s)$ and a v–generic polarization H. The Lagrangian fibration $\pi : M_H(v) \to |D|$, $F \mapsto \operatorname{supp}(F)$ as above is called a *Beauville–Mukai system of K3$^{[n]}$–type.*

Remark 4.4.13 More classically Beauville–Mukai systems arise from linear systems induced by a smooth curve in S in the following way, cf. [**Bea99**]. Let $C \subset S$ denote an irreducible smooth curve of genus n. Under the assumption that $\operatorname{Pic}(S)$ is generated by $\mathcal{O}_S(C)$ all curves in the linear system $|C|$ are reduced and irreducible. By Riemann–Roch it follows that $|C| = \mathbb{P}^n$. Let $\mathcal{C} \to \mathbb{P}^n$ denote the associated family of curves. For each d, the relative compactified Jacobian $\pi : X \coloneqq \overline{\operatorname{Pic}}^d(\mathcal{C}/\mathbb{P}^n) \to \mathbb{P}^n$ exists, see [**D'S79**, II, 1-4] or [**AK80**, Thm. 6.6].

Setting $v := (0, c_1(C), d+1-n)$ there is an identification of $\overline{\mathrm{Pic}}^d(\mathcal{C}/\mathbb{P}^n)$ with $M_H(v)$, see [**Muk84**, Ex. 0.5], given by the following map. For a pair $(\mathcal{C}_t, \mathcal{F})$ representing an element in X, consider the inclusion $\iota : \mathcal{C}_t \hookrightarrow S$. Then associate to it the element $\iota_\star \mathcal{F} \in M_H(v)$.

In particular one can see $M_H(v)$ as a generalization of the classical definition of a Beauville–Mukai system since the construction with the compactified Picard scheme only works if the linear system contains only reduced and irreducible curves.

With the statements of Appendix B.3 we can compute the polarization type of a Beauville–Mukai system of K3$^{[n]}$–type.

Theorem 4.4.14 *The Picard number of the generic smooth fiber of a Beauville–Mukai system $\pi : X \to |D|$ of K3$^{[n]}$–type equals one. In particular we have for its polarization type*

$$\underline{\mathrm{d}}(\pi) \;=\; (1, \ldots, 1)\,.$$

Proof: The first statement follows immediately from Theorem B.3.2 and Lemma B.3.1 (i). A special Kähler class ω on $M_H(v)$ with respect to a fiber F which is a Jacobian of a curve restricts to the unique primitive polarization on F since $\rho(F) = 1$. Then by Lemma B.3.1 (ii) this polarization is principal, hence $\underline{\mathrm{d}}(\pi) = \underline{\mathrm{d}}(\omega|_F) = (1, \ldots, 1)$ by Proposition 3.4.4. □

4.4.15. Beauville–Mukai systems of generalized Kummer type. The situation for an abelian surface S is slightly different. As before, let v be a primitive Mukai vector on S of the form $v = (0, c_1(D), s)$ where D is a big and nef divisor on S i.e. D is ample, cf. Remark 4.4.9. We set $2n := (D, D) - 2$. Note that we have $h^0(S, D) = \frac{1}{2}(D, D) = n + 1$, see also Remark 4.4.9. Choose a v–generic ample class H on S, hence $M_H(v)$ is an holomorphic symplectic manifold and the fiber $K_H(v)$ over $(0, 0)$ of the Albanese map $\mathrm{Alb}_v : M_H(v) \to S \times S^\vee$ is an irreducible holomorphic symplectic manifold of generalized Kummer n–type where $2n = (D, D) - 2$ by Theorem 4.2.22.

For simplicity we now fix a reference point $F_0 \in M_H(v)$ such that $\det(F_0) = \mathcal{O}_S(D)$. By [**Yos01**] the Albanese map $\mathrm{Alb}_v : M_H(v) \to S \times S^\vee$ with respect the reference point $F_0 \in M_H(v)$ can be written as

$$(\mathrm{Alb}_v)_{F_0} \;=\; \alpha \times \det_{F_0}$$

where $\det_{F_0} : M_H(v) \to \mathrm{Pic}^0(S) = S^\vee$ is defined as $\det_{F_0}(F) := \det(F) \otimes (\det(F_0))^{-1}$ and α can be defined as

$$\alpha(F) \;:=\; \sum c_2(F) \;:=\; \sum_i n_i x_i \tag{4.4.16}$$

where we view $c_2(F)$ in the Chow ring represented by the cycle $[\sum_i n_i x_i]$, see [**Yos01**, 4.1 ff.] and [**O'G14a**, p. 11].

Then the Albanese fiber $K_H(v) = (\mathrm{Alb}_v)^{-1}_{F_0}(0, 0)$ is an irreducible holomorphic symplectic manifold of dimension $2n$ by Theorem 4.2.22 and for $F \in K_H(v)$ the fitting support $\operatorname{supp}(F)$ is an element of the linear system $|D|$ by Lemma 4.4.6. This leads to the following commutative diagram

$$
\begin{tikzcd}
K_H(v) \arrow[r, hook] \arrow[d, "\pi"'] & M_H(v) \arrow[r, "(\mathrm{Alb}_v)_{F_0}"] \arrow[d, "\pi"'] & S \times S^\vee \arrow[d, "\mathrm{pr}_{S^\vee}"] \\
{|D|} \arrow[r, hook] & \{D\} \arrow[r] & S^\vee = \mathrm{Pic}^0(S)
\end{tikzcd}
\tag{4.4.17}
$$

where $\{D\} \to S^\vee$ is the map $C \mapsto \mathcal{O}_S(C) \otimes \det(F_0)^{-1}$. The induced map $K_H(v) \to |D|$ is a Lagrangian fibration by Matsushita's Theorem 2.1.3.

Definition 4.4.18 In the setting as above, the Lagrangian fibration

$$\pi : K_H(v) \longrightarrow |D|\,, \quad F \longmapsto \operatorname{supp}(F)$$

is called a *Beauville–Mukai system of generalized Kummer type.*

Consider a smooth curve $C \in |D|$, then the fiber of $M_H(v) \to \{D\}$ is given by the Jacobian $\mathrm{Jac}^d(C)$ of a certain degree d, see Remark 4.4.10. The restriction of the Albanese map $(\mathrm{Alb}_v)_{F_0} = \alpha_{F_0} \times \det_{F_0}$ to $\mathrm{Jac}^d(C) \subset M_H(v)$ is in the second component constant 0. Therefore, if we denote by $K^d(C) \subset \mathrm{Jac}^d(C)$ the fiber of $\pi : K_H(v) \to |D|$, we have an exact sequence

$$0 \longrightarrow K^d(C) \hookrightarrow \mathrm{Jac}^d(C) \xrightarrow{\alpha} S \tag{4.4.19}$$

where $\alpha = \mathrm{pr}_S \circ (\mathrm{Alb}_v)_{F_0}$ and the diagram

$$
\begin{tikzcd}
K_H(v) \arrow[r, hook] & M_H(v) \arrow[r, "(\mathrm{Alb}_v)_{F_0}"] & S \times S^\vee \\
K^d(C) \arrow[r, hook] \arrow[u, hook] & \mathrm{Jac}^d(C) \arrow[r, "\alpha"] \arrow[u, hook] & S \arrow[u, hook]
\end{tikzcd}
\tag{4.4.20}
$$

Lemma 4.4.21 *The map $\alpha = \mathrm{Jac}(\iota)$ above is the map induced by the inclusion $\iota : C \hookrightarrow S$ by the universal property of the Jacobian. More precisely α is given by*

$$\mathcal{O}_C(\textstyle\sum_i n_i x_i) \longmapsto \textstyle\sum_i n_i x_i\,.$$

In particular, $K^d(C)$ is the kernel of this map.

Proof: This follows from the definition of the map α, see (4.4.16). If $F \in \mathrm{Jac}^d(C) \subset M_H(v)$, then α takes on $\mathrm{Jac}^d(C)$ the form $\mathcal{O}_C(\sum_i n_i x_i) \mapsto \sum_i n_i x_i$ which is the map induced by ι and the universal property of the Jacobian, see subsection B.2.3. The second statement is obvious. □

By Lemma B.2.5 we know that we can see the dual $S^\vee = \mathrm{Pic}^0(S)$ as an abelian subvariety of $\mathrm{Jac}^d(C)$, as the pullback $\iota^\star : \mathrm{Pic}^0(S) \hookrightarrow \mathrm{Jac}(C) \cong \mathrm{Jac}^d(C)$ is an embedding. We conclude that we are in the situation of B.2.3 and therefore have the following.

Proposition 4.4.22 *In the situation above, $K^d(C)$ and $S^\vee$ are a pair of complementary abelian subvarieties in the principally polarized abelian variety* $\mathrm{Jac}^d(C)$ *in the sense of* section B.2.

Proof: Follows immediately from Lemma B.2.7. □

Further we can compute the polarization types of Beauville–Mukai systems of generalized Kummer type.

Theorem 4.4.23 *The Picard number of the generic smooth fiber of a Beauville–Mukai system $\pi : X \to |D|$ of generalized Kummer n–type equals one. In particular we have for its polarization type*

$$\underline{\mathrm{d}}(\pi) = (1, \ldots, 1, d_1, d_2)$$

where $\underline{\mathrm{d}}(D) = (d_1, d_2)$ is the type of the polarization defined by D.

Proof: Let us denote by $C \in |D|$ a generic smooth curve. The fiber $F = K(C) = K^d(C)$ of π over C is given as the kernel of the map $\mathrm{Jac}(\iota) : \mathrm{Jac}^d(C) \to S$, see (4.4.19), where $\iota : C \hookrightarrow S$ is the inclusion. We are therefore precisely in the situation of Theorem B.3.3 which states that $\rho(K(C)) = 1$ for the Picard number. Let $\omega \in \mathcal{K}_X$ denote a special Kähler class for the fiber $K(C)$. We are in the case of subsection B.2.3 and by Proposition B.2.14 the abelian subvariety $K(C)$ admits a polarization L of type $\underline{\mathrm{d}}(L) = (1, \ldots, 1, d_1, d_2)$. Since $\rho(K(C)) = 1$, we have $L = \omega|_{K(C)}$ as both are primitive. Therefore $\underline{\mathrm{d}}(\pi) = \underline{\mathrm{d}}(\omega|_F) = \underline{\mathrm{d}}(L) = (1, \ldots, 1, d_1, d_2)$ by Proposition 3.4.4. □

4.5. O'Grady's examples revisited

We consider again a projective holomorphic symplectic surface S. To construct O'Grady's exceptional examples [**O'G99**], [**O'G03**] one considers a non primitive Mukai vector v with an H–generic polarization. Therefore the moduli space $M_H(v)$ admits honest semistable sheaves which corresponds to singular points.

Recall the following terminology. Let X be a normal complex space such that the smooth part X^{reg} carries a holomorphic symplectic form σ. A *symplectic resolution* is a resolution $\phi : \widetilde{X} \to X$ such that $\phi^\star\sigma$ extends to a holomorphic symplectic form on $\widetilde{X}$.

The following result, which is stated in [**O'G14a**, Thm. 4.1], is a summary of results of several authors. But the main part is due to K. O'Grady.

Theorem 4.5.1 *On a projective holomorphic symplectic surface S let $v = kw$ with $k > 1$ be a non primitive Mukai vector and H a v–generic polarization. Then $M_H(v)$ is irreducible of dimension $2 + (v, v)$ and the smooth locus is given by the subset of stable sheaves. There exists a symplectic resolution $\phi : \widetilde{M_H}(v) \to M_H(v)$ if and only if $k = 2$ and $(w, w) = 2$. Further, the following holds.*

(*i*) *If S is K3, then $\widetilde{M}_H(2w)$ is irreducible holomorphic symplectic of dimension* 10 *with Betti number $b_2 = 24$.*

(*ii*) *If S is an abelian surface, then $\widetilde{K}_H(2w) := \phi^{-1}(K_H(2w))$ is irreducible holomorphic symplectic of dimension* 6 *with Betti number $b_2 = 8$.*

Originally, K. O'Grady proved that $M_H(2(1,0,-1))$ admits a symplectic resolution and that $\widetilde{M}_H(2(1,0,-1))$ (K3 case) and $\widetilde{K}_H(2(1,0,-1))$ (abelian surface case) are irreducible holomorphic symplectic with $b_2(\widetilde{K}_H(2(1,0,-1))) = 8$, cf. [**O'G99**], [**O'G03**]. A. Rapagnetta [**Rap08**] proved $b_2(\widetilde{M}_H(2(1,0,-1))) = 24$ in the K3 case. D. Kaledin, M. Lehn and C. Sorger [**KLS06**] proved the non existence of symplectic resolutions for $k > 2$ and $(w,w) > 2$.

4.5.2. Examples of Lagrangian fibrations on the O'Grady manifolds. In the setting as above we choose an ample divisor D on S with $(D,D) = 2$ and take a Mukai vector of the form

$$v \;:=\; (0, 2c_1(D), 2s) \;=\; 2(0, c_1(D), s)\,.$$

Note that we do not have much choice for a Mukai vector in consideration of defining a Lagrangian fibration on the moduli spaces. In the notation of Theorem 4.5.1, we have the map

$$\pi \;:\; \widetilde{M}_H(v) \xrightarrow{\phi} M_H(v) \xrightarrow{\text{supp}} \{2D\}$$

where in the K3 case we have $\{2D\} = |2D| = \mathbb{P}^5$ and the map is a Lagrangian fibration by Matsushita's Theorem 2.1.3.

In the abelian surface case and with the same choices as in subsection 4.4.15, in particular of a reference point $F_0 \in M_H(v)$ with $\det(F) = \mathcal{O}_S(2D)$, we have the Albanese fiber $K_H(v)$ with respect to this reference point and the diagram

$$\begin{tikzcd} \widetilde{M}_H(v) \arrow[r, "\phi"] & M_H(v) \arrow[r, "\text{supp}"] & \{2D\} \\ \widetilde{K}_H(v) \arrow[r, "\phi"] \arrow[u, hook] & K_H(v) \arrow[r, "\text{supp}"] \arrow[u, hook] & {|2D|} \arrow[u, hook] \end{tikzcd} \tag{4.5.3}$$

where the vertical arrows are inclusions. In thise case we have $h^0(S, 2D) = 4$ i.e. $|2D| = \mathbb{P}^3$. Again, the composition $\pi := \text{supp} \circ \phi : \widetilde{K}_H(v) \to |2D|$ is a Lagrangian fibration by Matsushita's Theorem 2.1.3.

Definition 4.5.4 We refer to the Lagrangian fibrations

$$\pi : \widetilde{M}_H(v) \to |2D| \quad \text{and} \quad \pi : \widetilde{K}_H(v) \to |2D|$$

as described above, as *Beauville–Mukai systems of O'Grady* 10*–type and O'Grady* 6*–type*, respectively.

Remark 4.5.5 Let $C \in |2H|$ denote a smooth element. The fiber of the morphism supp on $M_H(v)$ is by definition the Jacobian $\text{Jac}^d(C)$ where d is determined by v, see Remark 4.4.10. K. O'Grady showed, that the singularities of $M_H(v)$ are given by H–polystable[7] sheaves F, such that F is of the form $F = G_1 \oplus G_2$ where G_i are H–stable non isomorphic sheaves with $v(G_i) = w$ where $v = 2w$, see [**O'G99**] and for a good overview [**O'G14a**, 4.2]. This implies that elements in the Jacobian $\mathrm{Jac}^d(C) \subset M_H(v)$ do not belong to the singularities of $M_H(v)$. Therefore the fiber of $\phi \circ \mathrm{supp}$ is also given by $\mathrm{Jac}^d(C)$. This implies that the fibers of the Beauville–Mukai systems are $\pi^{-1}(C) = \mathrm{Jac}^d(C)$ for the K3 and $\pi^{-1}(C) = K^d(C)$ for the torus case.

Remark 4.5.5 enables us to compute the polarization types of the examples of O'Grady type fibrations above.

Theorem 4.5.6 *The polarization type of a Beauville–Mukai system* $\pi : \widetilde{M}_H(v) \to |2D|$ *of O'Grady* 10*–type is*

$$\underline{\mathrm{d}}(\pi) \;=\; (1,1,1,1,1)\,.$$

Proof: For a smooth curve $C \in |2D|$ the associated fiber is the Jacobian $\pi^{-1}(C) = \mathrm{Jac}^d(C)$ by Remark 4.5.5. By Theorem B.3.2 we can choose C such that $\rho(\pi^{-1}(C)) = 1$. The rest is similar to the proof of Theorem 4.4.14: A special Kähler class ω on $\widetilde{M}_H(v)$ with respect to the fiber $\pi^{-1}(C)$ restricts to the unique primitive polarization on $\pi^{-1}(C)$ since $\rho(\pi^{-1}(C)) = 1$. Then by Lemma B.3.1 (ii) this polarization is principal, hence $\underline{\mathrm{d}}(\pi) = \underline{\mathrm{d}}(\omega|_F) = (1, \ldots, 1)$ by Proposition 3.4.4. □

Theorem 4.5.7 *The polarization type of a Beauville–Mukai system* $\pi : \widetilde{K}_H(v) \to |2D|$ *of O'Grady* 6*–type is*

$$\underline{\mathrm{d}}(\pi) \;=\; (1,2,2)\,.$$

Proof: Let denote $C \in |2D|$ a smooth curve. The fiber $\pi^{-1}(C) = K(C) = K^d(C)$ of π over C is given as the kernel of the map $\mathrm{Jac}(\iota) : \mathrm{Jac}^d(C) \to S$, see (4.4.19), where $\iota : C \hookrightarrow S$ is the inclusion. We are therefore precisely in the situation of Theorem B.3.3 which states that $\rho(K(C)) = 1$ for the Picard number. Let ω denote a special Kähler class on $\widetilde{K}_H(v)$ for the fiber $\pi^{-1}(C) = K(C)$. Note that $\underline{\mathrm{d}}(2D) = (2,2)$ as D is a principal polarization on the abelian surface S. We are in the case of subsection B.2.3 and by Proposition B.2.14 the abelian subvariety $K(C)$ admits a polarization L of type $\underline{\mathrm{d}}(L) = (1,2,2)$. Since $\rho(K(C)) = 1$, we have $L = \omega|_{\pi^{-1}(C)}$ as both are primitive. Therefore $\underline{\mathrm{d}}(\pi) = \underline{\mathrm{d}}(\omega|_{\pi^{-1}(C)}) = \underline{\mathrm{d}}(L) = (1,2,2)$ by Proposition 3.4.4. □

These examples suggest that the polarization types of Lagrangian fibrations of O'Grady type are $(1,1,1,1,1)$ or $(1,2,2)$, respectively.

[7] Polystable means, that the graded sum (4.2.4) $\mathrm{gr}(F)$ of the Jordan–Hölder filtraion is isomorphic to F.

CHAPTER 5

Monodromy Invariants

In this chapter, the monodromy invariant for isotropic classes for generalized Kummer manifolds as stated in the introduction, cf. Theorem I.8 is constructed, see section 5.2. This is an analogue of E. Markman's monodromy invariant for the $\mathrm{K3}^{[n]}$ case, see subsection 5.2.10. This enables us finally to compute the polarization type of Lagrangian fibrations of generalized Kummer and $\mathrm{K3}^{[n]}$–type, see section 5.4.

5.1. Basic facts

In this section we recall basic facts about general monodromy invariants, as described in [**Mar13**, 5.3.].

Let X be an irreducible holomorphic symplectic manifold. Let $I(X) \subset H^2(X,\mathbb{Z})$ denote a monodromy invariant subset, i.e. $\mathrm{Mon}^2(X) \cdot I(X) \subset I(X)$ and Σ a set.

Definition 5.1.1 [**Mar13**, Def. 5.16] A *monodromy invariant* of the the pair (X,e), $e \in I(X)$, is a $\mathrm{Mon}^2(X)$–invariant map $\vartheta : I(X) \to \Sigma$ i.e. $\vartheta(ge) = \vartheta(e)$ for all $e \in I(X)$ and all $g \in \mathrm{Mon}^2(X)$. Further ϑ is called *faithful* if the induced map $\bar{\vartheta} : I(X)/\mathrm{Mon}^2(X) \to \Sigma$ is injective.

5.1.2. Induced monodromy invariant subset. Let X' denote another irreducible holomorphic symplectic manifold deformation equivalent to X. Let $P : H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ denote a parallel transport operator. Then we can define

$$I(X') \ := \ P(I(X))$$

to obtain a $\mathrm{Mon}^2(X')$ invariant subset $I(X')$ of $H^2(X',\mathbb{Z})$ induced by $I(X)$. Indeed this is well defined: if one has another parallel transport operator $P' : H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$, then $P'^{-1} \circ P$ is in $\mathrm{Mon}^2(X)$ hence $(P'^{-1} \circ P)(I(X)) = I(X)$ as $I(X)$ is $\mathrm{Mon}^2(X)$ invariant. Hence $P(I(X)) = P'(I(X))$.

Alternatively we could define

$$I(X') \ = \ \left\{ e' \in H^2(X',\mathbb{Z}) \mid \text{there exists } e \in I(X) \text{ such that } (X,e) \sim_{\mathrm{def}} (X',e') \right\} .$$

where the deformation equivalence of the pairs (X,e) and (X,e') is meant in the sense of Definition 1.3.10 as usual.

5.1.3. Induced monodromy invariant. Let X' be as above. If we have a monodromy invariant $\vartheta : I(X) \to \Sigma$ then we can obtain an induced monodromy invariant on X' which we also denote by $\vartheta : I(X') \to \Sigma$ by abuse of notation. If

$e' \in I(X')$ then there is a pair (X, e) deformation equivalent to (X', e') and we can define the induced monodromy invariant by

$$\vartheta(e') \ := \ \vartheta(e)\,.$$

Note that this is well defined as ϑ is $\mathrm{Mon}^2(X)$–invariant.

The following is a very important statement for the computation of polarization types of Lagrangian fibrations and is based on the Global Torelli Theorem, see [**Mar13**, 5.2 ff.].

Proposition 5.1.4 [**Mar13**, Lem. 5.17] *Let $\vartheta : I(X) \to \Sigma$ be a faithful monodromy invariant and let (X_i, e_i), $i = 1, 2$, denote two pairs with with X_i deformation equivalent to X and $e_i \in I(X_i)$.*

- *(i) $\vartheta(e_1) = \vartheta(e_2)$ if and only if (X_1, e_1) and (X_2, e_2) are deformation equivalent.*
- *(ii) If $\vartheta(e_1) = \vartheta(e_2)$ and $e_i = c_1(L_i)$ for holomorphic line bundles L_i on X_i and there exist Kähler classes ω_i on X_i such that $(\omega_i, e_i) > 0$, then (X_1, L_1) is deformation equivalent to (X_2, L_2).*

For effective isotropic classes, the requirements of the second statement of the Proposition above is always satisfied due to the following Lemma.

Lemma 5.1.5 *Let λ be a nontrivial isotropic class in the closure $\bar{\mathcal{C}}_X$ of the positive cone in $H^{1,1}(X, \mathbb{R})$ with X an arbitrary irreducible holomorphic symplectic manifold. Then the Beauville–Bogomolov quadratic form satisfies $(x, \lambda) > 0$ for every class x in the positive cone $\mathcal{C}_X$.*

Proof: Let $x \in \mathcal{C}_X$. As $\mathcal{C}_X$ is self–dual the cone coincides with its dual i.e. $\mathcal{C}_X = (\mathcal{C}_X)^\vee$. This means $(x, y) > 0$ for all $y \in \mathcal{C}_X$. Taking the closure of the positive cone, this condition changes to $(x, y) \geq 0$, in particular $(x, \lambda) \geq 0$. As $(x, x) > 0$ and the signature of the Beauville–Bogomolov form on $H^{1,1}(X, \mathbb{R})$ is $(1, b_2(X) - 3)$ (cf. [**GHJ03**, Cor. 23.11]) the orthogonal complement $x^\perp$ in $H^{1,1}(X, \mathbb{R})$ is a negative definite subspace. Therefore $\lambda \notin x^\perp$, otherwise $(\lambda, \lambda) < 0$. We conclude $(x, \lambda) > 0$. □

By definition the positive cone $\mathcal{C}_X$ contains the Kähler cone $\mathcal{K}_X$, therefore we always find Kähler classes as required in (ii) of Proposition 5.1.4, if the considered classes e_i are isotropic.

5.2. Monodromy invariants for isotropic classes

As we have seen in section 2.2, there is a close relation between Lagrangian fibrations and isotropic line bundles. We are therefore interested on monodromy invariants defined on the subset of isotropic classes of the second cohomology of an irreducible holomorphic symplectic manifold.

In this section a monodromy invariant for the isotropic classes on generalized Kummer manifolds, see subsection 5.2.5 is constructed in analogy of [**Mar11**, 2.], see subsection 5.2.10. First, we fix some notation for both cases.

Let X be a $\mathrm{K3}^{[n]}$–type or generalized Kummer type manifold of dimension $2n$. By Theorem 4.3.8 and Theorem 4.3.3 we have a canonical monodromy invariant $\mathrm{O}(\widetilde{\Lambda})$–orbit ι_X of primitive isometric embeddings from $\Lambda := H^2(X,\mathbb{Z})$ into the Mukai lattice $\widetilde{\Lambda}$ (4.2.8). Here, we use again the notation, that

$$\widetilde{\Lambda} \;=\; \begin{cases} \Lambda_{\mathrm{K3}} \oplus \mathrm{U} = E_8(-1)^{\oplus 2} \oplus U^{\oplus 4} & \text{if } \Lambda \text{ is the } \mathrm{K3}^{[n]} \text{ lattice,} \\ \mathrm{U}^{\oplus 4} & \text{if } \Lambda \text{ is the generalized Kummer lattice.} \end{cases}$$

Choose the following:

(*i*) A representative $\iota : \Lambda \hookrightarrow \widetilde{\Lambda}$ in ι_X.

(*ii*) A generator v of the sublattice $\iota(\Lambda)^\perp = \langle v \rangle$, cf. Remark 4.3.2.

Remark 5.2.1 The Kummer type lattice Λ has signature $(3,4)$, hence the orthogonal complement $\iota(\Lambda)^\perp$ is positive definite of rank one as the Mukai lattice $\widetilde{\Lambda} = \mathrm{U}^{\oplus 4}$ has signature $(4,4)$. Since the Gram discriminant of Λ is $-(2n{+}2)$ the Gram discriminant of $\iota(\Lambda)^\perp$ is $2n+2$, hence $(v,v) = 2n+2$. With the same argument, one gets $(v,v) = 2n-2$ for the $\mathrm{K3}^{[n]}$ case.

Furthermore, by [**Nik80**, Thm 1.14.4] there is a unique orbit of such primitive elements with square $2n+2$ (respectively $2n-2$ in the $\mathrm{K3}^{[n]}$ case) in $\widetilde{\Lambda}$. Since $\iota(\Lambda) = v^\perp$ we conclude that the action of $\mathrm{O}(\Lambda) \times \mathrm{O}(\widetilde{\Lambda})$ on $\mathrm{O}(\Lambda, \widetilde{\Lambda})$ is transitive.

For a primitive and isotropic element α in the Kummer type lattice Λ denote by $H(\alpha, \iota)$ the lattice defined by

$$H(\alpha,\iota) \;:=\; \mathrm{sat}\,\langle \iota(\alpha), v\rangle \;=\; \mathrm{sat}\,\langle \iota(\alpha), -v\rangle\;, \tag{5.2.2}$$

where sat denotes the saturation – the *saturation* of a sublattice L is the maximal sublattice of the same rank containing L, cf. Definition A.0.2. Further denote by

$$\vartheta(\alpha) \;:=\; [(H(\alpha,\iota), v)] \tag{5.2.3}$$

the isometry class of the pair $(H(\alpha,\iota), v)$ in the sense of Definition A.0.5.

Let d be a positive number such that d^2 divides $2n+2$ for the case that Λ is the generalized Kummer lattice or d^2 divides $2n-2$ if Λ is the $\mathrm{K3}^{[n]}$ lattice. Then define the lattice $L^{\Lambda}_{n,d}$ as $\mathbb{Z}^2$ with form

$$\frac{2n \pm 2}{d^2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{with} \quad \begin{cases} + & \text{if } \Lambda \text{ is the generalized Kummer lattice and} \\ - & \text{if } \Lambda \text{ is the } \mathrm{K3}^{[n]} \text{ lattice.} \end{cases} \tag{5.2.4}$$

We usually write just $L_{n,d}$ instead of $L^{\Lambda}_{n,d}$ if there is no confusion with the lattice Λ.

5.2.5. The generalized Kummer case. We now restrict to the generalized Kummer case, therefore X is a generalized Kummer type manifold and $\Lambda \cong H^2(X,\mathbb{Z})$

denotes the generalized Kummer lattice for this subsection, cf. (1.2.7).

The following Lemma is very similar to [**Mar14**, Lem. 2.5], see Theorem 5.2.11.

Lemma 5.2.6 *Let $\alpha \in \Lambda$ be a primitive isotropic class and set $d := \mathrm{Div}(\alpha)$.*

(*i*) *$\vartheta(\alpha)$ does not depend on the chosen representative $\iota \in \iota_X$.*

(*ii*) *For all $g \in \mathrm{Mon}^2(X)$ we have $\vartheta(g(\alpha)) = \vartheta(\alpha)$.*

(*iii*) *We can compose $\alpha \in \Lambda \cong \mathrm{U}^{\oplus 3} \oplus \langle -2n-2 \rangle$ as*

$$\alpha \;=\; d\xi + b\delta$$

where $\xi \in \mathrm{U}^{\oplus 3}$ is primitive, δ is the generator of $\langle -2n-2\rangle$ and $\gcd(d,b) = 1$. Further d^2 divides $n+1$.

(*iv*) *The lattice $H(\alpha,\iota)$ is isometric to the lattice $L_{n,d}$ defined in (5.2.4).*

(*v*) *There is an integer b, namely the one in (iii), such that $(\iota(\alpha) - bv)/d$ is integral (i.e. contained in $H(\alpha,\iota)$). Also any integer b with*
- *$\gcd(d,b) = 1$ and*
- *$(\iota(\alpha) - bv)/d$ is integral*

satisfies $\vartheta(\alpha) = [(L_{n,d}, (d,b))]$.

Proof: (*i*) Let $\iota_i \in \iota_X$, $i = 1,2$, be two representatives with $\iota_i(\Lambda)^\perp = \langle v_i \rangle$. Since the ι_i are in the same orbit ι_X there exists $\tilde{g} \in \mathrm{O}(\widetilde{\Lambda})$ such that $\tilde{g} \circ \iota_1 = \iota_2$ hence $\tilde{g}(\iota_1(\Lambda)) = \iota_2(\Lambda)$. We necessarily have $\tilde{g}(v_1) = \pm v_2$, otherwise we would have a contradiction to the bijectivity of $\tilde{g}$. We can assume $\tilde{g}(v_1) = v_2$ (otherwise take $-\tilde{g}$) then $\tilde{g}(\langle \iota_1(\alpha), v_1\rangle) = \langle \iota_2(\alpha), v_2 \rangle$ and the same holds for the saturation. Consequently $\tilde{g}$ gives the desired isometry of the pairs $(H(\alpha,\iota_i), v_i)$ hence $\vartheta(\alpha)$ does not depend on the chosen ι.

(*ii*) The orbit $\iota_X = \mathrm{O}(\widetilde{\Lambda})\iota$ is monodromy invariant that means we have a $\tilde{g} \in \mathrm{O}(\widetilde{\Lambda})$ such that $\tilde{g} \circ \iota = \iota \circ g$. With the same argument as in (i) we have $\tilde{g}(v) = \pm v$ (see Remark 4.3.2) and can assume $\tilde{g}(v) = v$. So $\tilde{g}$ defines an isometry between $\langle \iota(\alpha), v\rangle$ and $\langle \iota(g(\alpha)), v \rangle$ since $\tilde{g}(\iota(\alpha)) = \iota(g(\alpha))$ and in particular an isometry between the saturations $(H(\alpha,\iota), v)$ and $(H(g(\alpha),\iota), v)$, hence $\vartheta(\alpha) = \vartheta(g(\alpha))$.

(*iii*) Let δ be the generator of $\langle -2n-2 \rangle \subset \Lambda$. Then $\delta^\perp_\Lambda = \mathrm{U}^{\oplus 3}$. Since α is primitive we can write $\alpha = a\xi + b\delta$ such that $a > 0$ and $\xi \in \delta^\perp = \mathrm{U}^{\oplus 3}$ and $\gcd(a,b) = 1$. Then

$$0 \;=\; (\alpha,\alpha) \;=\; a^2(\xi,\xi) - (2n+2)b^2 \quad \Leftrightarrow \quad a^2(\xi,\xi) \;=\; (2n+2)b^2\,.$$

As (ξ,ξ) is even we get that a^2 divides $(n+1)$. Since δ is primitive we have $\mathrm{Div}(\delta) = 2n+2$ and $\mathrm{Div}(\xi) = 1$ since ξ is primitive and $\mathrm{U}^{\oplus 3}$ is unimodular, hence

$$d \;=\; \mathrm{Div}(\alpha) \;=\; \gcd(\mathrm{Div}(a\xi), \mathrm{Div}(b\delta)) \;=\; \gcd(a, (2n+2)b) \;=\; a\,.$$

(iv),(v) We use the same notation as in (iii). The lattice $\iota(\mathrm{U}^{\oplus 3})^\perp \subset \widetilde{\Lambda}$ is of rank 2 and contains $\iota(\delta)$ and v, hence it is the saturation of $\langle \iota(\delta), v\rangle$ as orthogonal complements are always saturated. As a complement of a unimodular lattice it is unimodular itself, hence it is the hyperbolic plane U. Consequently we can assume that $v = (1, n+1)$ and $\iota(\delta) = (1, -n-1)$. We have $\iota(\delta) - v = (2n+2)e$ where $e = (0, -1)$. Clearly e is isotropic. Then set

$$u \;:=\; \frac{1}{d}(bv - \iota(\alpha)) \;=\; -\iota(\xi) - \frac{b}{d}(2n+2)e\,.$$

Hence, the existence of such an integer b is proven.

As $\iota(\alpha) = -du + bv$ we have $\langle v, u\rangle \subset H(\alpha,\iota) := \operatorname{sat}\langle \iota(\alpha), v\rangle$. The complement $\delta_\Lambda^\perp = \mathrm{U}^{\oplus 3}$ is unimodular, hence we can find $\eta \in \delta_\Lambda^\perp$ such that $(\eta, \xi) = 1$ as $\xi \in \mathrm{U}^{\oplus 3}$ is primitive. For the intersection numbers we have

$$\begin{pmatrix} (v,e) & (v,\iota(\eta)) \\ (u,e) & (u,\iota(\eta)) \end{pmatrix} \;=\; \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Therefore the sublattice $\langle v, u\rangle \subset \widetilde{\Lambda}$ must be saturated, otherwise the determinant of the matrix above must be divisible by a nontrivial square. Consequently we have $H(\alpha,\iota) := \operatorname{sat}\langle \iota(\alpha), v\rangle = \langle v, u\rangle$.

Further $(v,u) = b\frac{2n+2}{d}$ and $(u,u) = b^2\frac{2n+2}{d^2}$. The Gram matrix G of $H(\alpha,\iota)$ with respect to the basis v, u is therefore

$$G \;=\; \frac{2n+2}{d^2}\begin{pmatrix} d^2 & bd \\ bd & b^2 \end{pmatrix} \;=\; \frac{2n+2}{d^2}\begin{pmatrix} d \\ b \end{pmatrix}\begin{pmatrix} d & b \end{pmatrix}.$$

Since $\gcd(d,b) = 1$ there are integers $i, j \in \mathbb{Z}$ with $id + jb = 1$. Set

$$A \;:=\; \begin{pmatrix} i & j \\ b & -d \end{pmatrix}.$$

This is an integral matrix with $A(d,b)^t = (1,0)^t$ and determinant -1, hence invertible over the integers. The Gram matrix with respect to the base change A is

$$A^t G A \;=\; \frac{2n+2}{d^2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore we have an isomorphism $L_{n,d} \cong H(\alpha,\iota)$ of lattices via $(x,y)^t \mapsto A(x,y)^t \cdot (v,u)$ where the product $\cdot$ is seen as a formal euclidean product. In particular (d,b) is mapped to v i.e. $\vartheta(\alpha) = [(L_{n,d}, (d,b))]$.

Now let b' be any integer satisfying the assumptions in (v). We know that (d,b') is primitive and that

$$u' \;:=\; \frac{1}{d}(b'v - \iota(\alpha))$$

is integral, therefore $u' - u = \frac{b'-b}{d}v$ is also integral. Since v is primitive, d must divide $b' - b$. Set $c := \frac{b'-b}{d} \in \mathbb{Z}$. Then

$$g_c := \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \in \mathrm{O}(L_{n,d})$$

is clearly an isometry of $L_{n,d}$ with $g_c(d,b)^t = (d, dc+b)^t = (d,b')^t$. Hence $\vartheta(\alpha) = [L_{n,d},(d,b)] = [L_{n,d},(d,b')]$. $\square$

Lemma 5.2.7 *The degenerate lattice $L_{n,d}$ embeds primitively and isometrically into $\widetilde{\Lambda} = \mathrm{U}^{\oplus 4}$ uniquely up to an isometry in $\mathrm{O}(\widetilde{\Lambda})$.*

Proof: Write $L := L_{n,d}$. The existence of such an embedding is clear, but follows also from [**Nik80**, Prop. 1.17.1]: there exists a primitive isometric embedding $L \hookrightarrow \widetilde{\Lambda}$ if and only if we can embed the quotient $L/\ker L$, where $\ker L$ denotes the null space of L, into some lattice of signature $(4-r, 4-r)$ where $r = \operatorname{rk}\ker L$. Since $\ker L = \langle (0,1)\rangle$ and $L/\ker L \cong \left\langle \frac{2n+2}{d^2}\right\rangle$, this is clearly possibly.

Also by [**Nik80**, Prop. 1.17.1] the isomorphism classes of primitive isometric embeddings $j : L \hookrightarrow \widetilde{\Lambda}$ are in one to one correspondence with isomorphism classes of induced primitive isometric embeddings

$$L/\ker L \hookrightarrow (\ker L)^{\perp}_{\widetilde{\Lambda}}/\ker L\,.$$

By Eichler's criterion A.0.6, we can assume that $j(0,1) = ((0,1),0,0,0)$ and $j(1,0) = (0,(\frac{n+1}{d^2},1),0,0)$. Then $(\ker L)^{\perp}_{\widetilde{\Lambda}} = \langle 0\rangle \oplus \mathrm{U}^{\oplus 3}$, hence $(\ker L)^{\perp}_{\widetilde{\Lambda}}/\ker L \cong \mathrm{U}^{\oplus 3}$. Now by Eichler's criterion A.0.6 there is up to an isometry in $\mathrm{O}(\widetilde{\Lambda})$ an unique way to embed $L/\ker L = \left\langle \frac{2n+2}{d^2}\right\rangle$ into $\mathrm{U}^{\oplus 3}$. $\square$

Lemma 5.2.8 *Let $\alpha \in \Lambda = \mathrm{U}^{\oplus 3} \oplus \langle -2n-2\rangle$ be a primitive isotropic element in the Kummer type lattice. Then there exists a $u \in \Lambda$ such that $(u,\alpha) = 0$ and $(u,u) = \pm 2$.*

Proof: Write $\alpha = \alpha_0 + \alpha_1$ with $\alpha_0 \in \mathrm{U}^{\oplus 3}$ and $\alpha \in \langle -2n-2\rangle$. The discriminant of $\mathrm{U}^{\oplus 3}$ is trivial since it's unimodular, hence by Eichler's criterion A.0.6 the $\mathrm{O}(\mathrm{U}^{\oplus 3})$–orbit of α_0 is determined by it's length $(\alpha_0,\alpha_0) = 2n+2$. So there exits an isometry $g \in \mathrm{O}(\mathrm{U}^{\oplus 3})$ such that $g(\alpha_0) = ((1,n+1),0,0) \in \mathrm{U}^{\oplus 3}$. Set $u := g^{-1}(0,0,(1,\pm 1)) \in \mathrm{U}^{\oplus 3} \subset \Lambda$. Then $(u,\alpha) = (u,\alpha_0) = 0$ and $(u,u) = ((1,\pm 1),(1,\pm 1)) = \pm 2$. $\square$

For a positive integer d let $I_d(X) \subset \Lambda = H^2(X,\mathbb{Z})$ denote the subset of primitive isotropic elements α such that $\mathrm{Div}(\alpha) = d$ which is clearly a $\mathrm{Mon}^2(X)$–invariant subset. Let $\Sigma_{n,d}$ denote the set of isometry classes of pairs (H,w) such that H is isometric to $L_{n,d}$ and $w \in H$ is a primitive class with $(w,w) = 2n+2$.

Theorem 5.2.9 *Let X be a Kummer type manifold of dimension $2n$ and d a positive integer such that d^2 divides $n+1$. The mapping*

$$\vartheta \;:\; I_d(X) \longrightarrow \Sigma_{n,d}\,, \quad \alpha \longmapsto \vartheta(\alpha) \;=\; [(H(\alpha,\iota),v)]$$

is a surjective faithful monodromy invariant of the manifold X.

Proof: By Lemma 5.2.6 $\vartheta : I_d(X) \longrightarrow \Sigma_{n,d}$ is well defined and $\mathrm{Mon}^2(X)$–invariant.

To show that ϑ is faithful i.e. that the induced map $\vartheta : I_d(X)/\mathrm{Mon}^2(X) \longrightarrow \Sigma_{n,d}$ is injective, we assume $\alpha_1,\alpha_2 \in I_d(X)$ with $\vartheta(\alpha_1) = \vartheta(\alpha_2)$, that means we have an isometry $g : H(\alpha_1, \iota) \to H(\alpha_2, \iota)$ with $g(v) = v$ where v is as usual a generator of $\iota(\Lambda)^\perp$.

We first show that both α_i lie in the same $\mathcal{W}(\Lambda)$–orbit, where the group $\mathcal{W}(\Lambda)$ was defined in Definition 1.5.7. We have $H(\alpha_i, \iota) \cong L_{n,d}$. By Lemma 5.2.7 there is up to an isometry in $\mathrm{O}(\widetilde{\Lambda})$ a unique way to embed $H(\alpha_i, \iota)$ isometrically and primitively into $\widetilde{\Lambda}$, hence we can extend g to an isometry $\tilde{g} \in \mathrm{O}(\widetilde{\Lambda})$. Since $v^\perp = \iota(\Lambda)$ we have in particular $\tilde{g}(\iota(\Lambda)) = \iota(\Lambda)$, i.e it makes sense to set $h := \iota^{-1} \circ \tilde{g} \circ \iota$ which is an isometry $h \in \mathrm{O}(\Lambda)$ such that $\iota \circ h = \tilde{g} \circ \iota$, hence h leaves the orbit $\iota_X = \mathrm{O}(\widetilde{\Lambda})\iota$ invariant and by Lemma 4.3.5 either $\mu = h$ or $\mu = -h$ is contained in the subgroup $\mathcal{W}(\Lambda)$ of orientation preserving isometries acting as ± 1 on the discriminant $\Lambda^\vee/\Lambda$. Choose μ such that it is in $\mathcal{W}(\Lambda)$. The null space of $H(\alpha_i, \iota) \subset \widetilde{\Lambda}$ is generated by $\iota(\alpha_i)$. Since $\tilde{g} \in \mathrm{O}(\widetilde{\Lambda})$ restricts to an isometry between $H(\alpha_1, \iota)$ and $H(\alpha_2, \iota)$ the null space of $H(\alpha_2, \iota)$ is generated by $\tilde{g}(\iota(\alpha_1)) = \iota(\pm h(\alpha_1)) = \iota(\mu(\alpha_1))$. So we have $\iota(\mu(\alpha_1)) = \pm\iota(\alpha_2)$, hence $\mu(\alpha_1) = \pm\alpha_2$. By Lemma 5.2.8 we can choose a $u \in \Lambda$ with $(u, \alpha_2) = 0$ and $(u, u) = +2$. Then the isometry $\rho_u \in \mathrm{O}(\Lambda)$ defined in Definition 1.5.9 i.e. $\rho_u(x) = -R_u(x) = -x + (u, x)u$ is contained in $\mathcal{W}(\Lambda)$, see Corollary 1.5.13 and Remark 1.5.10, and satisfies $\rho_u(\alpha_2) = -\alpha_2$, hence

$$\mathcal{W}(\Lambda)\alpha_1 \;=\; \mathcal{W}(\Lambda)(\pm\alpha_2) \;=\; \mathcal{W}(\Lambda)\alpha_2\,.$$

Now we show that $\mathcal{W}(\Lambda)\alpha = \mathrm{Mon}^2(X)\alpha$ for every primitive isotropic element $\alpha \in \Lambda$. Since $\mathrm{Mon}^2(X) \subset \mathcal{W}(\Lambda)$ is an index 2 subgroup by Corollary 1.5.13 we can write

$$\mathcal{W}(\Lambda) \;=\; \mathrm{Mon}^2(X) \;\cup\; \mathrm{Mon}^2(X)w$$

for every $w \in \mathcal{W}(\Lambda) \setminus \mathrm{Mon}^2(X)$. By Lemma 5.2.8 we have an element $u \in \Lambda$ with $(u, \alpha) = 0$ and $(u, u) = -2$. Then the reflection $\rho_u(x) = R_u(x) = x + (u, x)u$ of Λ (Definition 1.5.9) acts as $+1$ on the discriminant but has determinant -1, hence it is contained in $\mathcal{W}(\Lambda)$ but not in $\mathrm{Mon}^2(X)$, see again Corollary 1.5.13 and Remark 1.5.10. In particular $\rho_u(\alpha) = \alpha$ therefore

$$\begin{aligned}\mathcal{W}(\Lambda)\alpha \;&=\; \left(\mathrm{Mon}^2(X) \;\cup\; \mathrm{Mon}^2(X)\rho_u\right)\alpha \\ &=\; \mathrm{Mon}^2(X)\alpha \;\cup\; \mathrm{Mon}^2(X)\rho_u(\alpha) \;=\; \mathrm{Mon}^2(X)\alpha\,.\end{aligned}$$

For surjectivity, assume we have a class $[(L_{n,d}, w)] \in \Sigma_{n,d}$, i.e. $w \in L_{n,d}$ is primitive such that $(w, w) = 2n+2$. By Lemma 5.2.7 there exists a primitive isometric embedding $\iota_{n,d} : L_{n,d} \hookrightarrow \widetilde{\Lambda}$.

By Eichler's criterion A.0.6 we can assume that $\iota_{n,d}(w)$ is contained in a copy of U of $\widetilde{\Lambda} = \mathrm{U}^{\oplus 4}$. Then the lattice $\iota_{n,d}(w)^\perp \subset \widetilde{\Lambda} = \mathrm{U}^{\oplus 4}$ is of signature $(3, 4)$ and since $(w, w) = 2n + 2$ the complement $\iota_{n,d}(w)^\perp$ is isomorphic to $\Lambda \cong \mathrm{U}^{\oplus 3} \oplus \langle -2n - 2\rangle$.

The action of $\mathrm{O}(\Lambda) \times \mathrm{O}(\widetilde{\Lambda})$ on $\mathrm{O}(\Lambda, \widetilde{\Lambda})$ is transitive by Remark 5.2.1, hence the induced action of $\mathrm{O}(\Lambda)$ on the orbit set $\mathrm{O}(\Lambda, \widetilde{\Lambda})/\,\mathrm{O}(\widetilde{\Lambda})$ is also transitive. Hence, we can choose an isometry $g : \iota_{n,d}(w)^\perp \to \Lambda$ such that

$$\kappa \;:\; \Lambda \xrightarrow{g^{-1}} \iota_{n,d}(w)^\perp \subset \widetilde{\Lambda}$$

belongs to the monodromy invariant orbit $\iota_X = \mathrm{O}(\widetilde{\Lambda})\iota$. Recall from above that $(0,1) \in \ker L_{n,d}$ is the generator of $\ker L_{n,d}$. Clearly we have $(w,(0,1)) = 0$ in $L_{n,d}$ so we can set $\alpha := g(\iota_{n,d}(0,1))$. We can write

$$\alpha \;=\; a\xi + b\delta$$

where $\xi \in \mathrm{U}^{\oplus 3}$, $\delta \in \langle -2n-2\rangle$, $a > 0$ such that $\gcd(a,b) = 1$. As in the proof of Lemma 5.2.6 (iv) it follows that $a = \mathrm{Div}(\alpha)$. We have $\kappa(\Lambda)^\perp = \langle \iota_{n,d}(w)\rangle$ and $\kappa(\alpha) = \iota_{n,d}((0,1))$ and from Lemma 5.2.6 again

$$H(\alpha,\kappa) \;=\; \mathrm{sat}\,\langle \iota_{n,d}(0,1), \iota_{n,d}(w)\rangle \;\cong\; L_{n,a}\,,$$

where $\iota_{n,d}(w)$ is mapped to (a,b). The primitive element $w \in L_{n,d}$ is necessarily of the form $(\pm d, w_2)$ with $\gcd(d,w_2) = 1$. Over the rational numbers we have clearly $\iota_{n,d}(L_{n,d})_{\mathbb{Q}} = \langle \iota_{n,d}(0,1), \iota_{n,d}(w)\rangle_{\mathbb{Q}}$. As $\iota_{n,d}(L_{n,d})$ is saturated it follows that

$$\iota_{n,d}(L_{n,d}) \;=\; \mathrm{sat}\,\langle \iota_{n,d}(0,1), \iota_{n,d}(w)\rangle \;=\; H(\alpha,\kappa)\,.$$

Now we have an isometry

$$L_{n,d} \overset{\iota_{n,d}}{\hookrightarrow} H(\alpha,\kappa) \to L_{n,a}$$

where w is mapped to (a,b), hence $\mathrm{Div}(\alpha) = a = d$ i.e. $\alpha \in I_d(X)$ and $\vartheta(\alpha) = [(L_{n,d}, w)]$. ☐

5.2.10. The K3$^{[n]}$ case. We now consider the K3$^{[n]}$ case, which is due to E. Markman [**Mar14**, 2.5], therefore X is a K3$^{[n]}$–type manifold and $\Lambda := H^2(X,\mathbb{Z})$ denotes the K3$^{[n]}$ lattice for this subsection. And we have still fixed an representative ι of the monodromy invariant orbit ι_X and a v with $\iota(\Lambda)^\perp = \langle v\rangle \subset \widetilde{\Lambda}$, cf. (5.2.4).

As before, $I_d(X) \subset \Lambda = H^2(X,\mathbb{Z})$ denotes the $\mathrm{Mon}^2(X)$–invariant subset of primitive isotropic elements $\alpha \in \Lambda$ such that $\mathrm{Div}(\alpha) = d$. For a positiver integer d such that d^2 divides $n-1$, the target set $\Sigma_{n,d}$ is defined slightly different as the set of isometry classes of pairs (H,w) such that H is isometric to $L_{n,d} = L^{\Lambda}_{n,d}$ defined in (5.2.4) and $w \in H$ is a primitive class with $(w,w) = 2n-2$.

Theorem 5.2.11 (Markman, [**Mar14**, Lem. 2.5]) *Let X be a K3$^{[n]}$–type manifold of dimension $2n$ and d a positive integer such that d^2 divides $n-1$. The map*

$$\vartheta \;:\; I_d(X) \longrightarrow \Sigma_{n,d}\,, \quad \alpha \longmapsto \vartheta(\alpha) \;=\; [(H(\alpha,\iota), v)]$$

is a surjective faithful monodromy invariant of the manifold X. Any integer b with

- $\gcd(d,b) = 1$ *and*
- $(\iota(\alpha) - bv)/d$ *is integral*

satisfies $\vartheta(\alpha) = [(L_{n,d},(d,b))]$. Such an integer b always exists.

5.3. Beauville–Mukai systems in the moduli space

In this section we show that there are Beauville–Mukai systems in each connected component of the moduli of Lagrangian fibrations of $\mathrm{K3}^{[n]}$ and generalized Kummer type. We check this in terms of the monodromy invariants defined in the previous sections.

Please recall, that we mean by $L_{n,d}$ in the following two Propositions different lattices, cf. (5.2.4).

The proof of the following Proposition is similar to [**Mar14**, Ex. 3.1]. However, we give a detailed proof.

Proposition 5.3.1 *Let d be a positive integer such that d^2 divides $n+1$ and let b an integer satisfying $\gcd(d,b)=1$. Then there exists a Beauville–Mukai system $\pi : K_H(v) \to \mathbb{P}^n$ of generalized Kummer type and a primitive isotropic class $\alpha \in H^2(K_H(v),\mathbb{Z})$ such that the following holds.*

- (i) $\mathrm{Div}(\alpha) = d$,
- (ii) *the monodromy invariant $\vartheta(\alpha)$ is represented by $(L_{n,d},(d,b))$,*
- (iii) $c_1(\pi^\star \mathcal{O}_{\mathbb{P}^n}(1)) = \alpha$.
- (iv) *Its polarization type is given by $\underline{\mathrm{d}}(\pi) = (1,\ldots,1,d,\frac{n+1}{d})$.*

Proof: Let S be an abelian surface together with primitive ample line bundle L on S with $(L,L) = (2n+2)/d^2$. Set $\beta \coloneqq c_1(L)$ and let s be an integer such that $sb \equiv 1 \mod d$. Then $v \coloneqq (0,d\beta,s)$ is a Mukai vector. In particular v is primitive since β is primitive and $\gcd(d,s)=1$. Choose a v–generic ample class H. We have $(v,v) = d^2(\beta,\beta) = 2n+2$ hence $K_H(v) \subset M_H(v)$ is irreducible holomorphic symplectic of dimension $2n$ and we obtain a Beauville–Mukai system $\pi : M_H(v) \to |L^d|$ as described in section 4.4.15. We have Mukai's Hodge isometry

$$\Theta \;:\; v^\perp \longrightarrow H^2(M_H(v),\mathbb{Z}) \xrightarrow{r} H^2(K_H(v),\mathbb{Z})$$

as described in section 4.2.18. The map $r : H^2(M_H(v),\mathbb{Z}) \longrightarrow H^2(K_H(v),\mathbb{Z})$ is the restriction. Recall that definition of Θ needs the choice of a quasi–universal family of sheaves $\mathcal{E}$ on S of simlitude $\rho \in \mathbb{N}$.

Set $\alpha \coloneqq \Theta(0,0,1)$ which is clearly isotropic and define $\iota : H^2(K_H(v),\mathbb{Z}) \to H^\bullet(S,\mathbb{Z})$ to be Θ^{-1} composed with the inclusion $v^\perp \hookrightarrow H^\bullet(S,\mathbb{Z})$. Note that ι is a representative of the monodromy invariant orbit constructed in Theorem 4.3.8.

(i) An element (r,c,t) belongs to $v^\perp$ iff

$$0 \;=\; ((0,d\beta,s),(r,c,t)) \;=\; d(\beta,c) - rs \iff rs \;=\; d(\beta,c)\,.$$

Hence d divides r since $\gcd(d,s)=1$. Further we have $((0,0,1),(r,c,t)) = r$ for all $(r,c,t) \in v^\perp$ hence $\mathrm{Div}((0,0,1)) \geq d$. As the lattice of a two torus is $\mathrm{U}^{\oplus 3}$ i.e. in particular unimodular, we have $\mathrm{Div}_{H^2(S,\mathbb{Z})}(\beta) = 1$ in $H^2(S,\mathbb{Z})$. This implies that $\mathrm{Div}(\beta) = 1$ in $v^\perp$, hence we can find an element $c \in H^2(S,\mathbb{Z})$ such that $s = (c,\beta)$. Then $(d,c,0)$ is contained in $v^\perp$ and

$((0,0,1),(d,c,0)) = d$, hence

$$\mathrm{Div}(\alpha) \;=\; \mathrm{Div}(0,0,1) \;=\; d\,.$$

(*ii*) We have $\iota(\alpha) - bv = (0,0,1) - (0, bd\beta, bs) = (0, bd\beta, 1-bs)$ which is divisible by d since $sb \equiv 1 \mod d$ by assumption. By Lemma 5.2.6 (v) the monodromy invariant $\vartheta(\alpha)$ is represented by $(L_{n,d},(d,b))$.

(*iii*) Let $\omega = [p] \in H^4(S,\mathbb{Z})$ denote Poincare dual of a point $p \in S$. By our notation we have $\omega = (0,0,1) = \omega^\vee \in H^\bullet(S)$. Since S is an abelian surface, one has $\sqrt{\mathrm{Td}(S)} = 1$, see Example 4.2.11, hence $\sqrt{\mathrm{Td}(S)}\omega = \omega$. Note that $\mathcal{E}$ is a sheaf of rank zero, hence $\mathrm{ch}(\mathcal{E}) = \rho c_1(\mathcal{E}) + \xi = \rho[D] + \xi$ for some divisor D in $S \times M_H(v)$ and for some terms ξ of higher degree. Further $(\mathrm{pr}_S)^\star\omega = [p \times M_H(v)] \in H^4(S \times M_H(v),\mathbb{Z})$ and $[(\mathrm{pr}_{M_H(v)})_!(\xi \cdot [p \times M_H(v)])]_2 = 0$ due to degree reasons. Then we have

$$\begin{aligned}
\Theta(0,0,1) &\;=\; r\left((\mathrm{pr}_{M_H(v)})_!\,(D \cdot [p \times M_H(v)])\right)\\
&\;=\; r\left([F \in M_H(v) \mid p \in \mathrm{supp}(F)]\right)\\
&\;=\; [F \in K_H(v) \mid p \in \mathrm{supp}(F)]\\
&\;=\; \pi^\star[C \in |L^d| \mid p \in C]\\
&\;=\; \pi^\star c_1(\mathcal{O}_{|L^d|}(1)) \;=\; c_1(\pi^\star\mathcal{O}_{|L^d|}(1))
\end{aligned}$$

since $V := \left\{C \in |L^d| \mid p \in C\right\}$ is a hyperplane in a projective space, hence $[V] = c_1(\mathcal{O}_{|L^d|}(1))$.

(*iv*) This follows directly from Theorem 4.4.23 since $\underline{\mathrm{d}}(L) = (1, \frac{n+1}{d^2})$ by Lemma B.2.8 i.e. $\underline{\mathrm{d}}(dL) = (d, \frac{n+1}{d})$. □

The following Proposition is the $\mathrm{K3}^{[n]}$ version of Proposition 5.3.1. The proof is very similar to the one of Proposition 5.3.1. You can also find a detailed proof for the following Proposition in the author's paper [**Wie15**].

Proposition 5.3.2 [**Mar14**, Ex. 3.1] *Let d be a positive integer such that d^2 divides $n-1$ and let b an integer satisfying $\gcd(d,b) = 1$. Then there exists a Beauville–Mukai system $\pi : M_H(v) \to \mathbb{P}^n$ of $\mathrm{K3}^{[n]}$–type and a primitive isotropic class $\alpha \in H^2(M_H(v),\mathbb{Z})$ such that the following holds.*

(*i*) $\mathrm{Div}(\alpha) = d$,

(*ii*) *the monodromy invariant $\vartheta(\alpha)$ is represented by $(L_{n,d},(d,b))$,*

(*iii*) $c_1(\pi^\star\mathcal{O}_{\mathbb{P}^n}(1)) = \alpha$.

5.4. Computation of polarization types

We have now gathered everything to compute the polarization types of a Lagrangian fibration of $\mathrm{K3}^{[n]}$–type and generalized Kummer type.

Theorem 5.4.1 *Let $f : X \to \mathbb{P}^n$ be a Lagrangian fibration of $K3^{[n]}$–type or generalized Kummer type. Then for the polarization type $\underline{\mathrm{d}}(f)$ we have*

$$\underline{\mathrm{d}}(f) \;=\; \begin{cases} (1,\ldots,1) & \textit{in the } K3^{[n]} \textit{ case and} \\ (1,\ldots,1,d,\frac{n+1}{d}) & \textit{for the generalized Kummer case,} \end{cases}$$

where $d \coloneqq \mathrm{Div}(c_1(f^\star\mathcal{O}_{\mathbb{P}^n}(1)))$ denotes the divisibility of the associated element in the lattice $H^2(X,\mathbb{Z})$.

Proof: Let $f : X \to \mathbb{P}^n$ denote a Lagrangian fibration of $\mathrm{K3}^{[n]}$ or generalized Kummer type and set $L \coloneqq f^\star\mathcal{O}_{\mathbb{P}^n}(1)$. Then $\lambda \coloneqq c_1(L)$ is primitive by Corollary 2.2.16 and isotropic by Lemma 2.2.5 with respect to the Beauville–Bogomolov quadratic form. Let $d \coloneqq \mathrm{Div}(\lambda)$ denote the divisibility of λ. Consider the monodromy invariant $\vartheta : I_d(X) \to \Sigma_{n,d}$ as in Theorem 5.2.9 and Theorem 5.2.11, respectively. By Lemma 5.2.6 (v) and Theorem 5.2.11 there exists an integer b such that $\vartheta(\lambda)$ is represented by $(L_{n,d},(d,b))$ and we have $\gcd(d,b) = 1$.

By Theorem 5.3.2 and Theorem 5.3.1 we have a Beauville–Mukai system $\pi : X' \to \mathbb{P}^n$ of $\mathrm{K3}^{[n]}$ or generalized Kummer type, respectively, together with a primitive isotropic class $\alpha \in H^2(X',\mathbb{Z})$ such that $\mathrm{Div}(\alpha) = d$, $L' \coloneqq \pi^\star\mathcal{O}_{\mathbb{P}^n}(1)$ satisfies $c_1(L') = \alpha$ and $\vartheta(\alpha)$ is represented also by $(L_{n,d},(d,b))$ i.e. $\vartheta(\alpha) = \vartheta(\lambda)$.

Further by Lemma 5.1.5 we have $(\omega, L) > 0$ and $(\omega', L') > 0$ for Kähler classes ω on X and ω' on X' as L and L' are isotropic and nef, therefore are contained in $\bar{\mathcal{K}}_X \subset \bar{\mathcal{C}}_X$ and $\bar{\mathcal{K}}_{X'} \subset \bar{\mathcal{C}}_{X'}$ respectively. Hence we can apply Lemma 5.1.4 to see that the pairs (X,L) and (X',L') are deformation equivalent in the sense of Definition 1.3.10. By Proposition 2.4.8 there exist markings η and η' on X and X' respectively such that the pairs (X,η) and (X',η') are contained in the same connected component of the moduli of Lagrangian fibrations $\mathfrak{U}^\circ_{\lambda'}$ for a primitive isotropic class $\lambda' \coloneqq \eta(c_1(L)) \in \Lambda$. By Theorem 3.4.6, Theorem 5.3.2 and Theorem 5.3.1 we have

$$\underline{\mathrm{d}}(f) \;=\; \underline{\mathrm{d}}(\pi) \;=\; \begin{cases} (1,\ldots,1) & \text{in the } \mathrm{K3}^{[n]} \text{ case and} \\ (1,\ldots,1,d,\frac{n+1}{d}) & \text{for the generalized Kummer case} \end{cases}$$

which concludes the proof. □

APPENDIX A

Lattices

As the Beauville–Bogomolov-Fujiki form on the second cohomology $H^2(X,\mathbb{Z})$ of an irreducible holomorphic symplectic manifolds defines the structure of an abtract lattice, we recall some basic definitions and notations which are used frequently in this work.

Definition A.0.2 By a *lattice* or *abstract lattice* we mean a free abelian group $\Lambda \cong \mathbb{Z}^r$ of finite rank together with an integral bilinear form, usually denoted by $(\cdot,\cdot) : \Lambda \times \Lambda \to \mathbb{Z}$. Note that one can recover the bilinear form from its associated quadratic form $x \mapsto (x,x)$ as usual with the polarization identity.

(i) The positive number $r = \operatorname{rk}\Lambda$ is called the *rank* of Λ.

(ii) For $R \in \{\mathbb{Q},\mathbb{R},\mathbb{C}\}$ we denote the associated R–vector space by $\Lambda_R \coloneqq \Lambda \otimes_{\mathbb{Z}} R$. By abuse of notation, the induced bilinear form is also denoted by $(\cdot,\cdot)$.

(iii) The *signature* of Λ is the signature of $\Lambda_{\mathbb{R}}$.

(iv) Λ is called *even* if for all $x \in \Lambda$ the self intersection (x,x) for all $x \in \Lambda$ is even. Otherwise it is called *odd*. We mostly deal with even lattices in this work.

(v) Λ is called *(non)degenerate* if $(\cdot,\cdot)$ is (non)degenerate.

(vi) A *sublattice* is a subgroup $L \subset \Lambda$ which is a lattice with the restriction of $(\cdot,\cdot)$

(vii) An element $x \in \Lambda$ is called *primitive* if x is indivisible i.e. if $x = ky$ for some positive number k and $y \in \Lambda$, then $k = 1$.

($viii$) An element $x \in \Lambda$ is called *isotropic* if $(x,x) = 0$.

(ix) A sublattice L is called *primitive*, if the quotient Λ/L is torsion free.

(x) If $x_1,\ldots,x_n$ are in Λ, we denote the sublattice generated by those elements by

$$\langle x_1,\ldots,x_n\rangle\,.$$

(xi) The lattice $\mathbb{Z}e = \langle e\rangle$ generated by a single element e with $(e,e) = k$ is also denoted by $\langle k\rangle$.

(xii) The *saturation* of a sublattice L in Λ is the maximal sublattice $\operatorname{sat}(L)$ of the same rank $\operatorname{rk}\operatorname{sat}(L) = \operatorname{rk}\Lambda'$ of Λ such that $\operatorname{sat}(L)$ contains L.

($xiii$) The *dual* or *discriminant* of Λ is defined as $\Lambda^\vee = \operatorname{Hom}(\Lambda,\mathbb{Z})$. We have a canonical homomorphism $\Lambda \to \Lambda^\vee$ defined by $x \mapsto (x,\cdot)$.

(xiv) If $e_1,\ldots,e_r$ is a basis of Λ, then we call the matrix $G \coloneqq ((e_i,e_j))_{ij}$ the *Gram matrix* of Λ with respect to this basis. Its determinant $\det(G)$ is called *Gram determinant* or *Gram discriminant* and by standard linear algebra it is independent of the choice of the basis.

(xv) Λ is called *unimodular* if the Gram determinant is $\det(G) = \pm 1$. In this case, the natural map $\Lambda \to \Lambda^\vee$ is injective.

(xvi) The *null space* $\ker L$ of a sublattice L is the sublattice defined as the kernel $\ker(\Lambda \to \Lambda^\vee)$ of the canonical map.

($xvii$) The *divisibility* or the *divisor* of an element $x \in \Lambda$ is defined as

$$\mathrm{Div}(x) \;\coloneqq\; \max\left\{k \in \mathbb{N} \mid (x, \cdot)/k \text{ is an integral class in the dual } \Lambda^\vee\right\} .$$

Equivalently, $\mathrm{Div}(x)$ is the unique positive generator of the ideal $(x, L) = \mathrm{Div}(x)\mathbb{Z} \subset \mathbb{Z}$. Note that if the lattice is unimodular, then $\mathrm{Div}(x) = 1$ for every primitive element x.

The lattice which appears most frequently in this work is the unimodular hyperbolic lattice U of rank two which is $\mathbb{Z}^2$ with the form

$$\mathrm{U} \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{A.0.3}$$

It is well known that the second cohomology $H^2(S, \mathbb{Z})$ of a smooth complex surface is torsion free and the intersection form defines the structure of a lattice. For a K3 surface S this lattice is isomorphic to the *abstract K3 lattice*

$$\Lambda_{\mathrm{K3}} \;\coloneqq\; E_8(-1)^{\oplus 2} \oplus \mathrm{U}^{\oplus 3} , \tag{A.0.4}$$

see [**BHPV03**, p. 241, Prop. 3.2], where $E_8(-1)$ the negative definite root lattice of type E_8 which is defined in [**BHPV03**, p. 18].

Definition A.0.5 Let $\Lambda, \Lambda_1, \Lambda_2$ denote lattices and $e_i \in \Lambda_i$ elements.

(i) A homomorphism $g : \Lambda_1 \to \Lambda_2$ is called an *isometric* if $((f(x), f(x)) = (x, x)$ for all $x \in \Lambda_1$. Further g is called an *isometry* if g is isometric and an isomorphism.

(ii) The group of isometries $g : \Lambda \to \Lambda$ is denoted by $\mathrm{O}(\Lambda)$. As usual, such isometries have determinant ± 1.

(iii) An isometric homomorphism $g : \Lambda_1 \to \Lambda_2$ is called *primitive*, if the image $\mathrm{im}(g)$ is a primitive sublattice of Λ_2.

(iv) An *isomorphism* of the pairs (Λ_i, e_i) is an isometry $g : \Lambda_1 \to \Lambda_2$ such that $g(e_1) = e_2$.

Recall the following useful and well known criterion of M. Eichler.

Proposition A.0.6 (Eichler's criterion, [**Eic52**, 10.]) *Let Λ be an even lattice which contains two copies of the hyperbolic plane U. Then the $\mathrm{O}(\Lambda)$–orbit of a primitive element $x \in \Lambda$ is determined by its length (x, x) and the element $(x, \cdot)/\mathrm{Div}(x) \in \Lambda^\vee$.*

APPENDIX B

Abelian Varieties

We gather some useful statements about abelian varieties which are used in this thesis. To some readers the statements might be well known, for some they might not. They mostly follow from the general theory as explained in [**BL03**].

Apart from that, subsection B.2.3 and section B.3 are crucial for the main results of this thesis. Especially Lemma B.2.9 is important and the statement of it seems simple but the proof is not.

We fix some notation. Let A be a complex torus. We denote by $S^\vee \cong \operatorname{Pic}^0(S)$ the dual complex torus. An *isogeny* of complex tori is a surjective homomorphism $f : A \to B$ with finite kernel, in particular $\dim A = \dim B$.

If L is a line bundle on A, we denote by ϕ_L the homomorphism

$$\begin{aligned} \phi_L : A &\longrightarrow A^\vee \\ x &\longmapsto t_x^\star L \otimes L^{-1} \end{aligned} \tag{B.0.7}$$

see [**BL03**, 2.4], where $t_x : A \to A, a \mapsto a + x$ denotes translation by x.

Recall that the associated *Albanese torus* to a compact Kähler manifold X is defined as

$$\operatorname{Alb}(X) \;\coloneqq\; H^0(X, \Omega_X^1)^\vee / H_1(X, \mathbb{Z})$$

where we see a cycle $\gamma \in H_1(X, \mathbb{Z})$ via $\omega \mapsto \int_\gamma \omega$ as an element in $H^0(X, \Omega_X^1)^\vee$. If C is a smooth projective curve C, then the Albanese torus is the well known Jacobian $\operatorname{Jac}(C) \cong \operatorname{Pic}^0(C)$ of C. For a fixed point $x_0 \in X$, the Albanese map is defined as

$$\begin{aligned} \operatorname{Alb}_X \;: X &\longrightarrow \operatorname{Alb}(X) \\ x &\longmapsto \;[\omega \mapsto \int_{x_0}^x \omega] \end{aligned}$$

where the integral is computed over a path connecting x_0 and x.

B.1. Polarizations and their types

Let A be an abelian variety.

Definition B.1.1 A *polarization* of A is the first Chern class $c_1(L)$ of an ample line bundle L on A. Often one considers L itself as a polarization. The pair (A, L) is also called *polarized abelian variety.*

If L is a polarization on A, then the homomorphism ϕ_L in (B.0.7) is an isogeny, see [**BL03**, 2.4 ff.].

For every abelian variety A one has an identification

$$H^2(A,\mathbb{Z}) = \bigwedge^2 H_1(A,\mathbb{Z})^\vee , \tag{B.1.2}$$

which is induced by the canonical map $\bigwedge^2 H^1(A,\mathbb{Z}) \to H^2(A,\mathbb{Z})$ and $H_1(A,\mathbb{Z})^\vee = H^1(A,\mathbb{Z})$, see [**BL03**, Cor. 1.3.2].

If L is any line bundle on A, we can therefore interpret the first Chern class

$$c_1(L) \ : \ H_1(A,\mathbb{Z}) \otimes H_1(A,\mathbb{Z}) \longrightarrow \mathbb{Z}$$

as an alternating integral form on the lattice $H_1(X_t,\mathbb{Z})$.

By the elementary divisor theorem we can find a basis of $H_1(A,\mathbb{Z})$ for which $c_1(L)$ has the form

$$c_1(L) \ = \ \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$$

where $D = \mathrm{diag}(d_1, \dots d_n)$ is an integral diagonal matrix with $d_i \geq 0$ and $d_i | d_{i+1}$. Note that n is the dimension of A.

Definition B.1.3 (Polarization Type) The tuple of integers

$$\underline{\mathrm{d}}(c_1(L)) \ \coloneqq \ \underline{\mathrm{d}}(L) \ \coloneqq \ (d_1, \dots, d_n)$$

is called the *type* of the line bundle L. If L is a polarization (i.e. an ample line bundle) one also calls $\underline{\mathrm{d}}(L)$ its *polarization type*. If $\underline{\mathrm{d}}(L) = (1, \dots, 1)$, then L and $\underline{\mathrm{d}}(L)$ are called *principal*.

By [**BL03**, Prop. 4.5.2] for polarization L on A the associated alternating form (B.1.2) on $H_1(A,\mathbb{Z})$ to $c_1(L)$ is nondegenerate, therefore for $\underline{\mathrm{d}}(L) = (d_1, \dots, d_n)$ we have that $d_i > 0$ for all i.

Remark B.1.4 (i) The Jacobian $\mathrm{Jac}(C)$ of a smooth projective curve C has a canonical principal polarization, namely the *theta divisor*, see [**BL03**, Prop 11.1.2], which is usually denoted by Θ.

(ii) Is L a polarization on an abelian variety S with $\underline{\mathrm{d}}(L) = (d_1, \dots, d_n)$, then by [**BL03**, 14.4] there is a natural polarization L_δ on the dual $S^\vee$, called *dual polarization*, characterized by the following equivalent properties

(a) $\phi_L^\star L_\delta$ is algebraically equivalent to $L^{d_1 d_n}$, (b) $\phi_{L_\delta}\phi_L = d_1 d_n \mathrm{id}_S$.

Further the type is given by $\underline{\mathrm{d}}(L_\delta) = (d_1, \frac{d_1 d_n}{d_{n-1}}, \dots, \frac{d_1 d_n}{d_2}, d_n)$. If we are on an abelian surface, then obviously $\underline{\mathrm{d}}(L) = \underline{\mathrm{d}}(L_\delta)$.

B.2. Complementary abelian subvarieties

If M is an abelian variety and $A \subset M$ is an abelian subvariety and L a polarization on M, then one can define a so called *complementary subvariety* B to A (with respect to L). We only consider the case when $L = \Theta$ is a principal polarization [**BL03**, 12.1], for the more general setting see [**BL03**, 5.3].

We assume for this section, that Θ is a principal polarization, therefore we can identify M with its dual $M^\vee$ via the homomorphism ϕ_Θ. By [**BL03**, Prop. 1.2.6] for any polarization L the isogeny ϕ_L has always a $\mathbb{Q}$–inverse and we can define the $\mathbb{Q}$–endomorphism

$$g_A \coloneqq \iota \circ \phi^{-1}_{\iota^\star L} \circ \iota^\vee : M \otimes \mathbb{Q} \longrightarrow M \otimes \mathbb{Q}$$

where $\iota = \iota_A : A \hookrightarrow M$ denotes the inclusion. Choose a positive number m such that mg_A is an endomorphism of M. By [**BL03**, Prop. 12.1.3] we have

$$B \;\coloneqq\; (\ker(mg_A))_0 \subset M \;=\; \ker \iota^\vee \;\cong\; (A/B)^\vee\,, \tag{B.2.1}$$

where $(\ker(mg_A))_0$ denotes the identity component. Further B is an abelian subvariety of M called the *complementary subvariety* to A (with respect to L). Conversely, A is also the complementary subvariety to B and (A, B) is called a *pair of complementary subvarieties* in M.

Proposition B.2.2 [**BL03**, Cor. 12.1.5] *Let (A, B) be a pair of complementary abelian subvarieties in a principally polarized abelian variety (M, Θ) with $\dim A \geq \dim B = r$. Denote by ι_A and ι_B the inclusions of A and B into M respectively and assume $\underline{\mathrm{d}}(\iota_B^\star\Theta) = (d_1, \dots, d_r)$. Then $\underline{\mathrm{d}}(\iota_A^\star\Theta) = (1, \dots, 1, d_1, \dots, d_r)$.*

B.2.3. The case of a Jacobian. Let $\iota : C \hookrightarrow S$ be a smooth curve in an abelian surface S. Denote by Θ the principal polarization of the Jacobian $\mathrm{Jac}(C)$ of C and define $K(C) \coloneqq \ker \mathrm{Jac}(\iota)$ to be the kernel of the homomorphism $\mathrm{Jac}(\iota)$ induced by the inclusion $\iota : C \hookrightarrow S$ and by the universal property of the Jacobian [**BL03**, 11.4.1.], i.e. we have an exact sequence

$$K(C) \hookrightarrow \mathrm{Jac}(C) \overset{\mathrm{Jac}(\iota)}{\longrightarrow} S\,.$$

Using the several identifications of the dual and double dual, the dual of the pullback or the double pullback

$$(\iota^\star)^\vee = (\iota^\star)^\star \;:\; \mathrm{Jac}(C) = (\mathrm{Jac}(C))^\vee = \mathrm{Pic}^0(C) \longrightarrow S = (S^\vee)^\vee = \mathrm{Pic}^0(S^\vee)$$

is nothing but the map $\mathrm{Jac}(\iota)$. Of course, you can also see $\mathrm{Jac}(\iota)$ as the Albanese map induced by ι

$$\mathrm{Alb}(\iota) \;:\; \mathrm{Alb}(C) \longrightarrow \mathrm{Alb}(S) = S$$

if you identify $\mathrm{Alb}(C)$ and $\mathrm{Jac}(C)$, cf. [**BL03**, Prop. 11.11.6]. More concretely, the map $\mathrm{Jac}(\iota)$ viewed as a map $\mathrm{Pic}^0(C) \to S$ is

$$\mathcal{O}_C\left(\sum n_i x_i\right) \longmapsto \sum n_i x_i\,. \tag{B.2.4}$$

Indeed, for a given point c, denote by $\alpha_c : C \hookrightarrow \mathrm{Jac}(C)$, $x \mapsto \mathcal{O}_C(x - c)$ the Abel–Jacobi map. Then

$$\mathrm{Jac}(\iota)(\alpha_c(x)) \;=\; \mathrm{Jac}(\iota)(\mathcal{O}_C(x-c)) \;=\; x - c \;=\; t_{-c}(x)\,,$$

therefore it satisfies exactly the property of the unique morphism as described in [**BL03**, 11.4.1.].

We can see the dual $S^\vee$ as an abelian subvariety of $\mathrm{Jac}(C)$ in the following sense.

Lemma B.2.5 *The pullback morphism $\iota^\star : S^\vee \to \mathrm{Jac}(C)$ is an injection. Therefore $K(C)$ is connected.*

Proof: Let L be a line bundle on S with $L|_C = \mathcal{O}_C$. We have the standard exact sequence

$$0 \longrightarrow L \otimes \mathcal{O}_S(-C) \longrightarrow L \longrightarrow L|_C = \mathcal{O}_C \longrightarrow 0\,. \tag{B.2.6}$$

Since C is effective, the line bundle $L(-C) = L \otimes \mathcal{O}_S(-C)$ has no holomorphic sections i.e. $H^0(S, L(-C)) = 0$. In particular, $L(-C)$ cannot be ample (cf. [**BL03**, Prop. 4.5.2]), therefore the associated hermitian form of $c_1(L(-C))$ must have less then four positive eigenvalues. By [**BL03**, Lem. 3.5.1] we then have $H^1(S, L^\vee(-C)) = 0$. The long exact sequence of (B.2.6) shows that $h^0(S, L) = h^0(S, \mathcal{O}_C) = 1$ i.e. L has a holomorphic section s. Since $0 = c_1(L|_C) = [V(s)]$, the zero set $V(s)$ of s is empty i.e. $L = \mathcal{O}_S(V(s)) = \mathcal{O}_S(0) = \mathcal{O}_S$.

For the second statement identify $\mathrm{Jac}(C)^\vee = \mathrm{Jac}(C)$ via the principal polarization. We have the short exact sequence

$$0 \longrightarrow S^\vee \longrightarrow \mathrm{Jac}(C) \longrightarrow \mathrm{Jac}(C)/S^\vee \longrightarrow 0$$

and by [**BL03**, Prop. 2.4.2] the dual sequence

$$0 \longrightarrow (\mathrm{Jac}(C)/S^\vee)^\vee \longrightarrow \mathrm{Jac}(C) \longrightarrow S \to 0$$

is also exact. Hence $K(C) = \ker(\mathrm{Jac}(C) \to S) \cong (\mathrm{Jac}(C)/S^\vee)^\vee$ i.e. $K(C)$ is connected. □

In other words we have the following.

Lemma B.2.7 *The abelian subvarieties $K(C)$ and $S^\vee \overset{\iota^\star}{\hookrightarrow} \mathrm{Jac}(C)$ are a pair complementary abelian subvarieties of $\mathrm{Jac}(C)$.*

Proof: With the discussion above we have $K(C) = \ker(\iota^\star)^\vee$ which is exactly the definition as in (B.2.1) □

We are interested in the type of the polarization induced by Θ.

Lemma B.2.8 *Let L be a polarization on an abelian surface S of type $\underline{\mathrm{d}}(L) = (d_1, d_2)$. Then $h^0(S, L) = d_1 d_2$. If $C \in |L|$ is a not necessarily smooth curve, then we have for its arithmetic genus $g_a = d_1 d_2 + 1$. Further, if $c_1(L)$ is primitive and $(L, L) = 2d$, then $\underline{\mathrm{d}}(L) = (1, d)$.*

Proof: By the well known formula for the (arithmetic) genus, we have $g_a = 1 + \frac{1}{2}(C, C)$. By the geometric Riemann–Roch [**BL03**, 3.6 ff.] and since L is ample, we have

$$d_1 d_2 \;=\; \chi(L) \;=\; h^0(S, L) \;=\; \frac{1}{2}(C, C) \;=\; g_a - 1\,.$$

If $(L, L) = 2d$ and $c_1(L)$ is primitive, the equation above also shows $2d = (L, L) = 2d_1 d_2$. Since d_1 divides d_2 and (d_1, d_2) is primitive as $c_1(L)$ is primitive, we have $d_1 = 1$ i.e. $\underline{\mathrm{d}}(L) = (1, d)$. Also compare with Remark 4.4.9. □

Lemma B.2.9 *Let (S, L) denote a polarized abelian surface of type $\underline{\mathrm{d}}(L) = (d_1, d_2)$. Then for every smooth curve $\iota : C \hookrightarrow S$ with $C \in |L|$ we have that the restriction*

$$\Theta|_{S^\vee} \;:=\; (\iota^\star)^\star\Theta$$

is a polarization of type (d_1, d_2), where Θ denotes the principal polarization on the Jacobian $\mathrm{Jac}(C) = \mathrm{Pic}^0(C)$ and $(\iota^\star)^\star$ is viewed as a map $\mathrm{Pic}(\mathrm{Jac}(C)) \to \mathrm{Pic}(S^\vee)$. In particular, if the Picard number is $\rho(S) = 1$, then $\Theta|_{S^\vee} = L_\delta$ where the latter is the dual polarization on $S^\vee$ to L, cf. Remark B.1.4.

Proof: The proof is divided in three steps. In the first, we assume for the Picard number $\rho(S) = 1$ and show the existence of such a curve in $|L|$. In the second and still under the assumption $\rho(S) = 1$ it is shown that it holds for every smooth curve in $|L|$. In the third step we drop the restriction on the Picard number. We set $d := d_1d_2$.

- We first assume $\rho(S) = 1$ for the Picard number and prove the existence of such a curve $C \in |L|$. Since $\rho(S) = 1$ we have also $\rho(S^\vee) = 1$. Note that we have $\underline{\mathrm{d}}(L_\delta) = (d_1, d_2)$ by Remark B.1.4.

  Consider the isogeny $\phi_L : S \to S^\vee$. Then

$$\ker\phi_L \;\cong\; (\mathbb{Z}/d_1\mathbb{Z} \times \mathbb{Z}/d_1\mathbb{Z}) \oplus (\mathbb{Z}/d_2\mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z}) \tag{B.2.10}$$

  by [**BL03**, Lem. 3.1.4][1]. On $\ker\phi_L$ we have the alternating Weil pairing

$$e : \ker\phi_L \times \ker\phi_L \longrightarrow \mathbb{C}^\star$$

  see [**BL03**, p. 160], for the special case of an abelian surface see also [**BL03**, Ex. 6.7.3]. For $[x] = ([x_i]), [y] = ([y_i]) \in \ker\phi_L$ with respect to the isomorphism in (B.2.10), the pairing e can be calculated as

$$e([x],[y]) \;=\; \exp\left(\frac{2\pi\mathsf{i}}{d_1}(x_3y_1 - x_1y_3)\right)\cdot\exp\left(\frac{2\pi\mathsf{i}}{d_2}(x_4y_2 - x_2y_4)\right),$$

  see [**BL03**, Ex. 6.7.3].

  Choose a subgroup $G \subset \ker\phi_L$ such that $G \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z}$ and which is isotropic with respect to the pairing above.

  Then ϕ_L factorizes as

$$\begin{tikzcd} S \arrow[rr, "\phi_L"] \arrow[dr, "p"'] & & S^\vee \\ & S/G \arrow[ur, "p^\star"'] & \end{tikzcd}$$

  where p is the canonical projection which is a $d = d_1d_2$ to 1 map. As G is isotropic, by [**BL03**, Prop. 6.7.1] the action of G on S lifts to a free action of G on L, in particular we can define $L_0 := L/G \in \mathrm{Pic}(S/G)$. Since p is of degree d and $p^\star L_0 = L$, we have

$$2d \;=\; (L, L) \;=\; (p^\star L, p^\star L) \;=\; d(L_0, L_0)\,,$$

[1] In [**BL03**] they use the notation $K(L)$ for $\ker\phi_L$.

i.e. $(L_0, L_0) = 2$, therefore L_0 is a principal polarization on S/G. Hence, $H^0(S/G, L_0) = \mathbb{C}\sigma$ for a nontrivial section σ. Define the curve $C_0 := V(\sigma)$. Then C_0 is an element of $|L_0|$ and we claim that C_0 is smooth and irreducible.

Indeed, assume $C_0 = C_1 + C_2$. Since $\rho(S/G) = 1$ we have $C_1 = m_1 L_0$ and $C_2 = m_2 L_0$ with positive integers m_i. Then

$$2 \;=\; C_0^2 \;=\; (C_1 + C_2)^2 \;=\; (m_1 + m_2)^2 (L_0, L_0) \;=\; 2({m_1}^2 + m_2^2 + 2m_1 m_2) > 2\,,$$

which is absurd.

If C_0 is not smooth, then let $\nu : \widetilde{C}_0 \to C_0$ be its normalization. For its genus we have $g(\widetilde{C}_0) < g_a(C_0) = 2$. If $g(\widetilde{C}_0) = 0$, then $\widetilde{C}_0 = \mathbb{P}^1$ which is absurd, since $\nu : \mathbb{P}^1 \to C_0 \hookrightarrow S/G$ would be a non constant regular map which is not possible. If $g(\widetilde{C}_0) = 1$ then $\widetilde{C}_0$ would be an elliptic curve which can be seen as an abelian subvariety of S/G after a translation, if necessary. Then $\widetilde{C}_0$ has a complementary abelian subvariety in the sense as above. This would mean $\rho(S/G) \geq 2$ which contradicts $\rho(S/G) = 1$.

We conclude that C_0 is irreducible and smooth. In particular, C_0 is of genus 2 and by Lemma B.2.5, $(S/G) \cong (S/G)^\vee$ embeds into $\mathrm{Jac}(C_0)$. Both have the same dimension, hence $S/G \cong \mathrm{Jac}(C_0)$.

Set $C := p^{-1}(C_0)$. Then C is an element of $|L|$ as $L = p^\star L_0$ and is smooth as p is etale. It has to be connected with a similar argument as above. Assume $C = C_1 \cup C_2$ is a disjoint union. As $\rho(S) = 1$ we have $C_i = m_i L'$ for positive integers m_i where L' is the primitive part of L. Then

$$0 \;=\; (C_1, C_2) \;=\; (m_1 L', m_2 L') \;=\; 2m_1 m_2 \frac{d_2}{d_1} > 0\,,$$

which is absurd.

Hence, C is a connected smooth curve.

Denote by $\iota : C \hookrightarrow S$ the inclusion, by $q := p|_C = p \circ \iota : C \to C_0$ the induced d to 1 cover and by Θ_0 the principal polarization on $\mathrm{Jac}(C_0)$. Since $\rho(S) = 1$, also $\rho(S^\vee) = 1$ and $\rho(\mathrm{Jac}(C_0)) = 1$, so we have for the pullback $(p^\star)^\star L_\delta = k\Theta_0$ for some positive integer k. As $p^\star$ is surjective of degree d, taking the self intersection on both sides gives

$$2k^2 \;=\; (k\Theta_0, k\Theta_0) \;=\; ((p^\star)^\star L_\delta, (p^\star)^\star L_\delta) \;=\; d(L_\delta, L_\delta) \;=\; 2d^2$$

and hence $k = d$ i.e.

$$(p^\star)^\star L_\delta \;=\; d\Theta_0\,. \tag{B.2.11}$$

As $q = p \circ \iota$ we have that $q^\star$ is the map

$$q^\star : \mathrm{Jac}(C_0) \xrightarrow{p^\star} S^\vee \xrightarrow{\iota^\star} \mathrm{Jac}(C)\,.$$

Since $\rho(\mathrm{Jac}(C_0)) = 1$, we have $(q^\star)^\star\Theta = a\Theta_0$ for some positive integer a. By [**BL03**, Lem. 12.3.1] we have that $(q^\star)^\star\Theta$ is algebraically equivalent to

$d\Theta_0$. Therefore $ac_1(\Theta_0) = c_1((q^\star)^\star\Theta) = dc_1(\Theta_0)$, hence $a = d$ i.e.

$$(q^\star)^\star\Theta \;=\; d\Theta_0\,. \tag{B.2.12}$$

Finally write $(\iota^\star)^\star\Theta = bL_\delta$ for some positive integer b. We have

$$d\Theta_0 \overset{(B.2.12)}{=} (q^\star)^\star\Theta \;=\; (\iota^\star\circ p^\star)^\star\Theta \;=\; (p^\star)^\star(\iota^\star)^\star\Theta \;=\; (p^\star)^\star(bL_\delta) \overset{(B.2.11)}{=} bd\Theta_0\,,$$

hence $b = 1$ i.e. $(\iota^\star)^\star\Theta = L_\delta$.

- We show that the statement holds for every element in $|L|$ but still assume $\rho(S) = 1$ for the Picard number.

  Consider the open and connected set $U \subset |L| \cong \mathbb{P}^{d-1}$ such that every element in U corresponds to a smooth curve in S. Let $\mathcal{C} \to U$ be the associated family of smooth curves. We can take the relative Jacobian

  $$\pi_k : X^k := \mathrm{Pic}^k(\mathcal{C}/U) \longrightarrow U$$

  of degree $k \in \mathbb{Z}$ of it, cf. Remark 4.4.13.

  By Lemma B.2.8 the genus of $\mathcal{C}_t$ is $g = d+1$. By considering the image of $(X^d)^{(d)} \to X^d$, $(x_1,\dots,x_d) \mapsto \sum_i x_i$ which is a divisor in X^d, we obtain a line bundle $\mathcal{M} \in \mathrm{Pic}(X^d)$ such that $\mathcal{M}_t := \mathcal{M}|_{X_t^d}$ is the natural polarization on $X_t^d = \mathrm{Pic}^d(\mathcal{C}_t)$.

  Locally we can identify X^d with X^0, say $X_V^d = \pi_d^{-1}(V) \cong X_V^0 = \pi_0^{-1}(V)$ where $V \subset U \subset |L|$ is chosen connected, by twisting with a line bundle Q^V on $\pi_d^{-1}(V)$ which has degree $-d$ on the fibers $\mathcal{C}_t$ for $t \in V$. Then we obtain on a line bundle $\mathcal{L}^V = \mathcal{M}\otimes Q^V$ on X_V^0, such that $\mathcal{L}_t^V := \mathcal{L}^V|_{X_t^0}$ is the principal polarization on $X_t^0 = \mathrm{Jac}(\mathcal{C}_t)$ for $t \in V$. Let $\iota_t : \mathcal{C}_t \hookrightarrow S$ denote the inclusion. Then the self intersection $m_V : V \to \mathbb{Z}$

  $$m_V(t) \;:=\; \Big((\iota_t^\star)^\star\mathcal{L}_t^V, (\iota_t^\star)^\star\mathcal{L}_t^V\Big) \tag{B.2.13}$$

  of $(\iota_t^\star)^\star\mathcal{L}_t^V$ is a continuous and integer valued function, therefore must be constant as V is chosen connected.

  By the first part we know that there is an element $t_0 \in U$ such that the statement for the curve $\mathcal{C}_{t_0}$ holds. For arbitrary $t_N \in U$, choose a path γ from t_0 to t_N in U. By the discussion above, we can cover γ with finitely many connected open sets $V_0,\dots,V_N$ such that $t_0 \in V_0$ and $t_N \in V_N$ and we have elements $t_i \in V_i \cap V_{i+1}$ for $i = 1,\dots N-1$. Then the self intersections m_{V_i} and $m_{V_{i+1}}$ must coincide on $V_i \cap V_{i+1}$.

  By assumption we have

  $$m_{V_0}(t_0) \;=\; (L_\delta, L_\delta) \;=\; 2d$$

  i.e. $m_{V_0} \equiv 2d$. Assume $(\iota_{t_N}^\star)^\star\mathcal{L}_{t_N} = kL_\delta$ for some positive integer k. Then

  $$2d = m_{V_0}(t_0) \;= m_{V_1}(t_1) = \ldots = m_{V_N}(t_N) = k^2 2d$$

  i.e. $k = 1$.

- We now consider the general case i.e. let S be with arbitrary Picard number. We have an universal family $p : \mathcal{X} \to \mathfrak{h}_2$ of (d_1, d_2)–polarized abelian surfaces over Siegel's upper half plane $\mathfrak{h}_2$, see [**BL03**, 8.7]. Let $\mathcal{N}$ denote the line bundle on $\mathcal{X}$ such that $\mathcal{N}_s := \mathcal{N}|_{\mathcal{X}_s}$ is the (d_1, d_2) polarization on $\mathcal{X}_s$ for $s \in \mathfrak{h}_2$.

  For each $s \in \mathfrak{h}_2$ let $U_s \subset |\mathcal{N}_s| \cong \mathbb{P}^{d-1}$ be the open set such that all elements in U_s corresponds to smooth curves in $\mathcal{X}_s$. Let $U \subset \mathbb{P}^{d-1}$ denote the open and connected subset such that for every for every $(s,t) \in \mathfrak{h}_2 \times U$ the point $t \in \mathbb{P}^{d-1} \cong |\mathcal{N}_s|$ corresponds to an element in U_s. In particular it corresponds to a smooth curve $\mathcal{C}^s_t$ in $\mathcal{X}_s$. Let $\iota_{s,t} : \mathcal{C}^s_t \hookrightarrow \mathcal{X}_s$ denote the inclusion.

  From the second step of the proof we know that for each $(s,t) \in \mathfrak{h}_2 \times \mathbb{P}^{d-1}$ we can find a neighbourhood $V_{s,t} \subset U_s$ of t and a relative principal polarization $\mathcal{L}^{s,t}$ on $\mathrm{Pic}^0(\mathcal{C}^s/|U_s)|_{V_{s,t}}$ where $\mathcal{C}^s \to U_s$ denotes the associated family of smooth curves to U_s.

  We can define the map

  $$\varphi : \mathfrak{h}_2 \times U \longrightarrow \mathbb{Z}^2\,, \quad (s,t) \longmapsto \underline{\mathrm{d}}\left((\iota^\star_{s,t})^\star \mathcal{L}^{s,t}_t\right)$$

  for the case that $(s,t) \in \mathfrak{h}_2 \times V_{s,t}$. This is well defined and continuous, therefore must be constant as U is connected. It is well known, see [**BL03**, 8.11, (1)], that the generic abelian surface has endomorphism ring $\mathrm{End} = \mathbb{Z}$ i.e. has Picard number $\rho = 1$, by Lemma B.3.1. Therefore the statement proven in the second step applies for a generic element $(s_0, t_0) \in \mathfrak{h}_2 \times U$ i.e. $\varphi(s_0, t_0) = (d_1, d_2) \equiv \varphi$.

  For our original situation this means that the type of $\Theta|_{S^\vee} = (\iota^\star)^\star\Theta$ is $\underline{\mathrm{d}}(\Theta|_{S^\vee}) = (d_1, d_2)$ for arbitrary (d_1, d_2)–polarized (S, L). $\square$

An immediate consequence of Lemma B.2.9 and Proposition B.2.2 is the following.

Proposition B.2.14 *Let (S, L) denote a polarized abelian surface of type $\underline{\mathrm{d}}(L) = (d_1, d_2)$. Then for every smooth curve $C \in |L|$, we have that the type of the restriction of the principal polarization Θ of $\mathrm{Jac}(C)$ to $K(C)$ is*

$$\underline{\mathrm{d}}(\Theta|_{K(C)}) \;=\; (1, \ldots, 1, d_1, d_2)\,.$$

Proof: By Lemma B.2.9 the restriction $\Theta|_{S^\vee}$ is a polarization of type $\underline{\mathrm{d}}(\Theta|_{S^\vee}) = (d_1, d_2)$. By Proposition B.2.2 the type of $\Theta|_{K(C)}$ is $\underline{\mathrm{d}}(\Theta|_{K(C)}) = (1, \ldots, 1, d_1, d_2)$. $\square$

B.3. Picard numbers of abelian varieties with less endomorphisms

Lemma B.3.1 *Let A be an abelian variety.*

(*i*) *If $\mathrm{End}(A) = \mathbb{Z}$ then its Picard number is $\rho(A) = 1$.*

(*ii*) *If* $A = \mathrm{Jac}(C)$ *is a Jacobian of a smooth curve* C *and* $\rho(A) = 1$ *then the primitive polarization* Θ *on* A *is principal i.e.* $\underline{\mathrm{d}}(\Theta) = (1, \dots, 1)$.

Proof: (*i*) By [**BL03**, Prop. 5.2.1] there is an isomorphism $\mathrm{NS}(A) \otimes \mathbb{Q} \cong V$ where V is a $\mathbb{Q}$–subspace of $\mathrm{End}(A) \otimes \mathbb{Q}$. The latter has dimension 1 by assumption hence $\rho(A) = \dim_{\mathbb{Q}} \mathrm{NS}(A) \otimes \mathbb{Q} = 1$.

(*ii*) It is well known that on every Jacobian of a curve there exists a primitive principal polarization, see [**BL03**, Prop 11.1.2]. Since $\rho(A) = 1$ it must be unique. □

We now consider Jacobians of curves which are contained in linear systems defined on K3 or abelian surfaces.

Theorem B.3.2 [**CvdG92**, Thm. 1.1, Cor. 1.2] *Let* S *be a projective* $K3$ *surface and* V *a linear system on it. If* C *is a general element of* V *and* $\mathrm{Jac}(C)$ *the Jacobian of* C *then* $\mathrm{End}(\mathrm{Jac}(C)) = \mathbb{Z}$ *and therefore* $\rho(\mathrm{Jac}(C)) = 1$.

Proof: This follows directly from [**CvdG92**, Thm. 1.1, Cor. 1.2] since K3 surfaces satisfy $H^1(S, \mathcal{O}_S) = 0$. One has to note that the condition in [**CvdG92**, Thm. 1.1] that V defines a birational map from S to its image can be dropped because the authors only use this to conclude that the pullback morphism

$$\mathrm{Pic}^0(S) \longrightarrow \mathrm{Pic}^0(C) = \mathrm{Jac}(C)$$

has finite kernel, see [**CvdG92**, p. 35, 2.II.]. Since S is K3 we have $\mathrm{Pic}^0(S) = 0$ and so this condition is satisfied. □

Theorem B.3.3 [**CvdG92**, 3.B.] *Let* S *be an abelian surface,* $\iota : C \hookrightarrow S$ *a smooth curve and let denote by*

$$K(C) := \ker(\mathrm{Jac}(C) \to S) \subset \mathrm{Jac}(C)$$

the kernel of the map $\mathrm{Jac}(\iota)$ *induced by the inclusion and the universal property of the Jacobian, as described in* subsection B.2.3. *Then* $\mathrm{End}(K(C)) = \mathbb{Z}$, *therefore we have for the Picard number* $\rho(K(C)) = 1$.

Proof: We know by Lemma B.2.5 that $K(C)$ is connected i.e. a honest abelian subvariety of $\mathrm{Jac}(C)$. Again, the requirement in [**CvdG92**, 2.II.] that $|C|$ defines a birational map on its image can be dropped, since the authors only use this to conclude that the map $\iota^\star : S^\vee \to \mathrm{Jac}(C)$ has finite kernel. In our setting this is the case by Lemma B.2.5. Then by [**CvdG92**, 3.B.] we have $\mathrm{End}(K(C)) = \mathbb{Z}$, hence $\rho(K(C)) = 1$ for the Picard number by Lemma B.3.1. □

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